A FOURIER METHOD OF DETERMINING THE SPEED FLUCTUATION COEFFICIENT OF A GENERAL SLIDER-CRANK MECHANISM

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In this paper a method is presented which enables the solution of the equation of motion of the slider-crank mechanism. Expansion of functions into Fourier series is used in this method, whereby the solution of the equation of motion is also obtained in the form of Fourier series. This method enables also the determination of the coefficient of speed fluctuation of the slider-crank mechanism. The method creates the basis for an analysis of the influence of various factors, e.g. the working conditions or the construction parameters of the system, on the coefficient of speed fluctuation.

1 INTRODUCTION

The slider-crank mechanism is one of the most frequently applied planar linkages in engineering [1]. It is a special configuration of the four-bar linkage with a slider replacing an infinitely long output link. The most popular application of this mechanism is the internal combustion engine, wherein the input force is the gas pressure on the piston. The same mechanism is widely used in agricultural and food-processing machines as well as in packing machines. In all these machines, but especially in agricultural machines, the occurrence of variable resistance of relatively high values creates a significant problem. This causes considerable fluctuation in the motion of the whole system. The coefficient of speed fluctuation is one measure of such fluctuations.

In order to calculate the coefficient of speed fluctuation (which is a criterion for assessing the performance of many machines) it is necessary to determine) the maximum and minimum values of the machine speed, either by calculation or measurement. Measurements can be done only on a real machine after its manufacture and it is undoubtedly a great advantage to be able to calculate the speed maxima during design. To do so, the equation of motion of the mechanism must be solved. This is a typical dynamics problem, in the category of "forward dynamics" problems [2]. It may also be described as "time-response" analysis, where the geometry, mass and inertia of the mechanism are known functions of position, as are the external loads, driving forces and torques. Time response analysis produces kinematic information about the linkage (including velocities of different links) as functions of position or time.

The problem discussed in this paper is a typical dynamic case of time-response analysis. It requires the solution of the equation of motion of the slider-crank mechanism, which is a non-linear second order differential equation. Methods exist to solve such equations numerically. Among the many available numerical integration techniques, one of the most widely used is the Runge-Kutta numerical analysis routine [7,8,9]. Although this method was originally invented for first-order equations, it is possible to apply it to higher-order systems. For the Runge-Kutta method to be applied for such a system, the higher-order equation is first transformed into a serious of first-order equations. These are then solved one at a time, using the results of each previous integration as the input for the next one.

However, a problem with the application of the Runge-Kutta method, as for any other numerical method, to the equations of motion of a mechanism is that the initial conditions may vary. Such methods require that the solution process be repeated for each set of conditions.. This poses a general difficulty in applying the analysis to a practical mechanism. The present paper addresses the initial value problem by introducing a method of solving the equation of motion based on the method of small parameters and on Fourier expansions.

2 EQUATION OF MOTION OF THE SLIDER-CRANK MECHANISM

In order to determine the physical model of the slider-crank mechanism (Fig. 1) the following assumptions have been made:

- clearances are absent in kinematic pairs,
- the performance of slider-crank does not affect the drive,
- link masses are concentrated at their centres of gravity.

6. links are rigid.

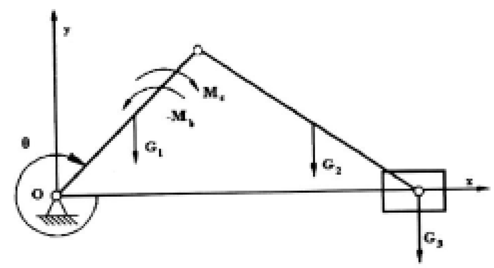

Figure 1: Physical model of a slider-crank mechanism.

Using Lagrange equations, the following equation of motion of the slider-crank mechanism was derived:

$$I\frac{d^2\theta}{dt^2}+\frac{1}{2}\left(\frac{dI}{d\theta}\right)(\theta)\left(\frac{d\theta}{dt}\right)^2=M_c+M_b(\theta) \qquad (1)$$

where,

θ - is the angle of the crank rotation,

t - is the time,

$$I(\theta)=\frac{1}{3}m_1 l_1^2+\frac{1}{12}m_2 l_1^2\frac{\cos^2\theta}{1-l^2\sin^2\theta}+ +m_2 l_1^2\left[\left(\sin\theta+\frac{l\sin 2\theta}{4\sqrt{1-l^2\sin^2\theta}}\right)^2+\frac{1}{4}\cos^2\theta\right]+ +m_3 l_1^2\left(\sin\theta+\frac{l\sin 2\theta}{2\sqrt{1-l^2\sin^2\theta}}\right)^2 \qquad (2)$$

$$M_b(\theta)=\frac{1}{2}l_1\left(G_1+G_2\right)\cos\theta+ +P\,l_1\left(\sin\theta+\frac{l\sin 2\theta}{2\sqrt{1-l^2\sin^2\theta}}\right) \qquad (3)$$

m_i - the mass of the link (i = 1, 2, 3);

G_i - the weight of the link (i = 1, 2, 3);

l_i - the length of the link (i = 1, 2),

$l = l_1 / l_2$

M_c - the driving moment,

P - the force acting on the piston (generally periodic with period 2π).

3 SOLUTION OF THE EQUATION OF MOTION

The equation of motion of the slider-crank mechanism is a non-linear ordinary differential equation. It is inconvenient to solve this equation by analytical methods because of the complicated form of the equation itself and of the coefficients occurring in it. The method based on the expansions of functions into Fourier series [3] is introduced in this paper.

The equation of motion (1) could be also written in the following form:

$$I(\theta)\ddot{\theta}+\frac{1}{2}I'(\theta)\dot{\theta}^2=M_c+M_b(\theta) \qquad (4)$$

where,

$$I'(\theta)=\frac{dI}{d\theta}\quad \dot{\theta}=\frac{d\theta}{dt}\quad \ddot{\theta}=\frac{d^2\theta}{dt^2}$$

θ is the generalised co-ordinate (angle of crank rotation), I(θ) is the mass moment of inertia referred to the crank (it is determined by the expression (2)), M_c is the driving torque (assumed as a constant value), $M_b(\theta)$ is the braking torque referred to the crank (determined by the relation (3)). The expressions I(θ) and $M_b(\theta)$ are periodic functions of the variable θ; their periods are 2π. These functions can be presented in the following form [3]:

$$I(\theta)=I_o+\tilde{I}(\theta) \qquad (5)$$

$$M_b(\theta)=M_{b_o}+\tilde{M}_b(\theta) \qquad (6)$$

where,

$$I_o=\frac{1}{2\pi}\int_0^{2\pi}I(\theta)\,d\theta \qquad (7)$$

$$M_{b_o}=\frac{1}{2\pi}\int_0^{2\pi}M_b(\theta)\,d\theta \qquad (8)$$

Applying the expressions (5) and (6), equation (4) can be written in the following form:

$$I_o\ddot{\theta}-M_c-M_{b_o}=-\tilde{I}(\theta)\ddot{\theta}-\frac{1}{2}\tilde{I}'(\theta)\dot{\theta}^2+\tilde{M}_b(\theta) \qquad (9)$$

where expressions evidently depending on the co-ordinate θ are arranged on the right hand side of the equation. They determine the deviations of the functions I(θ) and $M_b(\theta)$ from the mean values of I_o and M_{bo}, i.e. they are expressions causing motion fluctuation of the system in steady state. Assuming that motion fluctuation is a small value, a small parameter (δ) can be introduced on the right side of the equation [4,5,6] giving,

$$I_o\ddot{\theta}-M_c-M_{b_o}=\delta\left[-\tilde{I}(\theta)\ddot{\theta}-\frac{1}{2}\tilde{I}'(\theta)\dot{\theta}^2+\tilde{M}_b(\theta)\right] \qquad (10)$$

Changing the variables,

$$\dot{\theta}=\omega \quad and \quad \ddot{\theta}=\omega\frac{d\omega}{d\theta}=\omega\,\omega'$$

the equation (11) was obtained:

$$I_o \omega \omega' - M_c - M_{b_o} =$$
$$= \delta \left[-\tilde{I}(\theta)\omega\omega' - \frac{1}{2}\tilde{I}'(\theta)\omega^2 + \tilde{M}_b(\theta) \right] \quad (11)$$

The above equation is a first order non-linear differential equation, where the independent variable is the angle of crank rotation θ, and the unknown is the angular velocity ω. The solution of this equation will enable the determination of the $\omega(\theta)$ function. In the steady state the solution of $\omega(\theta)$ is a periodic function with period 2π or integer multiple of 2π. The solution will be assumed to be of the following form [4,5,6]:

$$\omega(\theta) = \omega_0 + \delta\omega_1(\theta) + \delta^2\omega_2(\theta) + \dots \quad (12)$$

where, ω_o is the constant mean crank angular velocity resulting from the drive, $\omega_i(\theta)$ are periodic functions of the period 2π (i=1, 2, ...). Substituting (12) into equation (11) will give a general equation in the following form.

$$I_o(\omega_o + \delta\omega_1 + \delta^2\omega_2 + \dots)(\delta\omega_1' + \delta^2\omega_2' + \dots) - M_c - M_{bo} =$$
$$\delta\Big[-\tilde{I}(\theta)(\omega_o + \delta\omega_1 + \delta^2\omega_2 + \dots)(\delta\omega_1' + \delta^2\omega_2' + \dots) +$$
$$-\frac{1}{2}\tilde{I}'(\theta)(\omega_o + \delta\omega_1 + \delta^2\omega_2 + \dots)^2 + \tilde{M}_b(\theta)\Big]$$

By grouping the expressions of the same power of small parameter and using only the elements up to the δ square, the following expressions are obtained:

$$\delta^0 : -M_c - M_{b_o} = 0 \quad (13a)$$

$$\delta^1 : I_o\omega_o\omega_1' = -\frac{1}{2}\tilde{I}'(\theta)\omega_o^2 + \tilde{M}_b(\theta) \quad (13b)$$

$$\delta^2 : I_o\omega_o\omega_2' = -I_o\omega_1\omega_1' - \tilde{I}(\theta)\omega_o\omega_1' + -\tilde{I}'(\theta)\omega_o\omega_1 \quad (13c)$$

Equation (13a) enables the determination of the driving torque M_c, that should be applied to the crank to overcome the load:

$$M_c = -M_{b_o} \quad (14)$$

Equations (13b) and (13c) constitute a set of equations enabling the determination of successive approximate solutions $\omega_1(\theta)$ and $\omega_2(\theta)$ to the function $\omega(\theta)$.

In order to solve the equation which determines the first approximation, the right side of equation (13b) was expanded into a Fourier series:

$$-\tilde{I}'(\theta)\omega_o^2 + \tilde{M}_b(\theta) = \sum_{k=1}^{\infty}\left(L_{c,k}\cos k\theta + L_{s,k}\sin k\theta\right) \quad (15)$$

(the constant term of the expansion equals zero). Through equations (13b) and (15) the following was obtained:

$$I_o\omega_o\omega_1' = \sum_{k=1}^{\infty}\left(L_{c,k}\cos k\theta + L_{s,k}\sin k\theta\right) \quad (16)$$

where,

$$L_{c,k} = \frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{2}\tilde{I}'(\theta)\omega_o^2 + \tilde{M}_b(\theta)\right]\cos k\theta \, d\theta \quad (17a)$$

$$L_{s,k} = \frac{1}{\pi}\int_0^{2\pi}\left[-\frac{1}{2}\tilde{I}'(\theta)\omega_o^2 + \tilde{M}_b(\theta)\right]\sin k\theta \, d\theta \quad (17b)$$

Expression (16) allows us to determine the periodic solution with the period 2π of equation (13b):

$$\omega_1(\theta) = \sum_{k=1}^{\infty}\left(\frac{L_{c,k}}{k I_o \omega_o}\sin k\theta - \frac{L_{s,k}}{k I_o \omega_o}\cos k\theta\right) \quad (18)$$

Equation (13c) can be solved similarly. The right hand side of the equation was expanded into a Fourier series:

$$I_o\omega_o\omega_1' - \tilde{I}(\theta)\omega_o\omega_1' - \tilde{I}'(\theta)\omega_o\omega_1 =$$
$$= \sum_{k=1}^{\infty}\left(N_{c,k}\cos k\theta + N_{s,k}\sin k\theta\right) \quad (19)$$

In this case the constant expression is also equal to zero, and, because of the periodicity ω_1 and $I(\theta)$ the following relations occur:

$$\frac{1}{2\pi}\int_0^{2\pi}\omega_1\omega_1' d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\left(\frac{\omega_1^2}{2}\right) = 0$$
$$\frac{1}{2\pi}\int_0^{2\pi}\left[\tilde{I}(\theta)\omega_1' + \tilde{I}'(\theta)\omega_1\right]d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\left[\tilde{I}(\theta)\omega_1\right] = 0$$

Periodic solution with 2π period of equation (13c) has the following form:

$$\omega_2(\theta) = \sum_{k=1}^{\infty}\left(\frac{N_{c,k}}{k I_o \omega_o}\sin k\theta - \frac{N_{s,k}}{k I_o \omega_o}\cos k\theta\right) \quad (20)$$

Substituting in equation (12) the value of the small parameter $\delta = 1$ [4,5,6] and taking into consideration the expressions determining $\omega_1(\theta)$ and $\omega_2(\theta)$, i.e. equations (18) and (20), the final formula was obtained which shows the dependence of angular velocity on the angle of rotation:

$$\omega(\theta) = \omega_o + \sum_{k=1}^{\infty}\left(\frac{L_{c,k}N_{c,k}}{k I_o \omega_o}\sin k\theta - \frac{L_{s,k}N_{s,k}}{k I_o \omega_o}\cos k\theta\right) \quad (21)$$

In the above calculation only the two first approximations of angular velocity $\omega_1(\theta)$ and $\omega_2(\theta)$ were taken into account. If greater accuracy is required, formulae can be developed analogously, determining subsequent approximations connected with higher powers of the small parameter δ.

In formula (21) an infinite sum of expressions occurs, which are the expansions into the Fourier series of appropriate functions. The accuracy of the calculations also depends on the number of expressions in this sum that are taken into account in specific calculations.

It should be stressed that for expansion of a function into the Fourier series it is not necessary to know the mathematical relation determining this function. It is enough to know the values of this function at some points. Thus it is possible to use the Runge's scheme for determination of the expansion coefficients. This makes it possible to carry out the above analysis in the case when no mathematical formula determining the braking torque $M_b(\theta)$ is available. This has a great significance in machines for which it is difficult to give an analytical formula for the function $M_b(\theta)$.

4 DETERMINATION OF THE COEFFICIENT OF SPEED FLUCTUATION

As it appears from the formula determining the relation between the angular velocity and the angle of rotation (21), the velocity changes around the mean value ω_o during a stable cycle motion. The coefficient of speed fluctuation is determined by the formula [1]:

$$C_s = \frac{\omega_{max} - \omega_{min}}{\omega_o} \qquad (22)$$

where, ω_{max} is the value of maximum velocity during the working cycle, ω_{min} is the value of minimum velocity, ω_o is the mean value. Thus for the determination of the coefficient of speed fluctuation the maximal and minimal values of the angular velocity should be determined. This may be done by comparing the values of angular velocities for different angles of rotation obtained by the formula (21). It is obvious that a computer should be used to perform such calculations.

5 DISCUSSION

The method here described may now be compared with less general numerical methods of solving the equation of motion of a slider-crank mechanism to obtain its coefficient of fluctuation. It is quite obvious that the mechanism will be in periodic motion with certain mean values of velocity or acceleration during its cycle but the initial conditions are usually not known. The "zero" initial conditions cannot be used, since the analysis is done for the steady-state situation and not for the initial acceleration or final deceleration of the system. The method that avoids this problem is a method of small parameters, by which the solution is sought around a certain value. The case presented seemed to be ideal for this particular method. The steady mean value of the crank angular velocity resulting from the drive is usually known, and so it is required only to look for deviations from this mean value (equation (12)). This method leads to a set of equations presented as (13) in this paper. Equations (13b) and (13c) are non-linear first order differential equations and a numerical routine may be used in order to solve them. The Runge-Kutta method might be used at this point, taking the result of integration of equation (13b) as the input to equation (13c)). However, expansion of functions into Fourier series provides a more straightforward method that converts these equations into a simple form easy to integrate. Once this expansion is done the solution is easy to obtain. The method presented overcomes the difficulty with the initial conditions of the mechanical system in a steady-state situation and also avoids the laborious numerical integration of the differential equations.

6 CONCLUSIONS

A method for the solution of the equation of motion of slider-crank mechanisms has been presented. The method uses the concept of small parameters and also requires the expansion of functions by Fourier series. Hence, the relation between the angular velocity and the angle of rotation of the crank, which is the solution of the equation of motion, is obtained in the form of a Fourier series. It is thus possible to calculate the coefficient of speed fluctuation for any slider-crank mechanism. A computer is needed for the application of this method. The method creates the basis for an analysis of the influence of various factors, e.g. the working conditions or the construction parameters of the system, on the coefficient of speed fluctuation.

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