ON THE OSCILLATORY FLOW PAST AN INFINITE VERTICAL POROUS PLATE I - MEAN FLOW

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Two-dimensional flow of an incompressible viscous fluid past an infinite, porous plate in a porous medium is considered for the unsteady flow with the following conditions: (1) the suction velocity normal to the plate is constant, (2) the free stream velocity oscillates in time about a constant mean, (3) the temperature of the plate is kept constant, (4) the difference between the temperature of the plate and the free stream is considerably large causing the free convection currents. Approximate solutions for the coupled non-linear equations are obtained for velocity and temperature. Expressions for the mean velocity, mean temperature and the mean skin friction are derived. The effects of the Grashof number G, the Prandtl number P, the Eckert number E and the Darcy number σ, on the mean motion of air and water are studied. It is found that the mean velocity, for air, increases due to more cooling ($G > 0$) of the plate by the free convection currents, for any value of σ, when E is constant. In the presence of heated ($G < 0$) plate, the mean velocity is negative for $\sigma = 0$ near the boundary layer and as σ increases the mean velocity gradually decreases in magnitude and becomes positive with further increase in σ. Heating and cooling have separate effects on the mean velocity for fluids for any value of σ with large Prandtl numbers. In the case of heating of the plate, the mean velocity increases whereas in the case of cooling of the plate, the mean velocity decreases with increasing Prandtl number.

1. INTRODUCTION

From the technological point of view oscillatory flow is always important, for it has many practical applications. Such a study was initiated by Lighthill [1] who studied a two-dimensional flow of an incompressible viscous fluid. By assuming that a regular fluctuating flow is superimposed on the mean steady boundary-layer flow, he solved the problem by the momentum method. Stuart [2] extended this idea to study a two-dimensional flow past an infinite porous plate when the free stream oscillates in-time about a constant mean, where he assumed that there is no heat transfer between the plate and the fluid in deriving the temperature field, which is only one of the possible cases of physical situation. Soundalgekar [3,4] discussed the other case of physical situation, that is, when the difference between the plate temperature and the free stream temperature is apparently large so as to cause the free convection currents to the flow in the boundary layer and the free stream velocity is also oscillating in time about a constant mean in the direction of the flow, then how is the flow field near a porous infinite, vertical plate with constant suction affected by the free convection currents? He assumed that (1) the plate temperature oscillates in time about a constant mean, (2) the free convective currents are present in the boundary layer and (3) the flow is very slow and hence viscous dissipative effects are negligible. After having solved the problem, he observed that the temperature field was not at all affected by the free convective currents, which is not always true since in the case of fluids with high Prandtl number, viscous dissipative heat is always present even in slow motion. This led him to study the effects of free convection currents on the oscillatory type of boundary layer flow past an infinite vertical plate with constant suction and the plate temperature differing from the free stream temperature. He also studied the effects of heating or cooling of the plate and those of greater viscous dissipative heat.

In the present paper, the author has made an attempt to seek the influence of the Darcy number σ on these oscillatory types of boundary layer flows in porous medium.

The work is done by mathematical analysis that is presented under suitable assumptions for the mean velocity, mean temperature and the mean shearing stress.

2. MATHEMATICAL FORMULATION

A two-dimensional, unsteady flow of an incompressible, viscous fluid past an infinite, porous plate with constant suction in a porous medium is considered. The x' - axis is taken along the vertical, infinite plate, in vertical direction, which is the direction of flow and z' - axis is taken normal to the plate. The governing equations are

Equations of momentum:

$$\rho'\left(\frac{\partial u'}{\partial t'} + w'\frac{\partial u'}{\partial z'}\right) = -\frac{\partial p'}{\partial x'} - \rho' g_x + \mu\frac{\partial^2 u'}{\partial z'^2} - \frac{\mu}{\kappa}u' \tag{1}$$

$$\frac{\partial w'}{\partial t'} = -\frac{1}{\rho'}\frac{\partial p'}{\partial z'} \tag{2}$$

Equation of continuity:

$$\frac{\partial w'}{\partial z'} = 0 \tag{3}$$

Equation of energy:

$$\rho' C_P \left(\frac{\partial T'}{\partial t'} + w'\frac{\partial T'}{\partial z'}\right) = K\frac{\partial^2 T'}{\partial z'^2} + \mu\left(\frac{\partial u'}{\partial z'}\right)^2 + \frac{\mu}{\kappa}u'^2 \tag{4}$$

where u' and w' are components of velocity in x' and z' directions respectively, ρ' the density of the fluid in the boundary layer, g_x the acceleration due to gravity, p' the pressure, μ the coefficient of viscosity, κ the permeability of the medium, C_P the specific heat at constant pressure, T' the temperature of the fluid and K the thermal conductivity. The boundary conditions are

$$u' = 0,\quad T' = T_w' \quad \text{at } z' = 0$$

$$u' = U'(t'),\quad T' = T_\infty' \quad \text{as } z' \to \infty \tag{5}$$

where T_w' is the temperature of the plate, T_∞' the temperature of the fluid in the free stream and U' the free stream velocity. In the free stream, from (1), we get

$$\rho'\frac{\partial U'}{\partial t'} = -\frac{\partial p'}{\partial x'} - \rho_\infty' g_x - \frac{\mu}{\kappa}U' \tag{6}$$

where ρ_∞' is the density of the fluid in the free stream. Eliminating $-\frac{\partial p'}{\partial x'}$ from (1) and (6), and using the equation of state

$$g_x(\rho_\infty' - \rho') = g_x\beta\rho'(T' - T_\infty') \tag{7}$$

where β is the coefficient of volume expansion, we get

$$\frac{\partial u'}{\partial t'} + w'\frac{\partial u'}{\partial z'} = \frac{\partial U'}{\partial t'} + g_x\beta(T' - T_\infty') + \nu\frac{\partial^2 u'}{\partial z'^2} - \frac{\nu}{\kappa}(u' - U') \tag{8}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Assuming constant suction velocity w_o at the plate, equation (3) integrates to

$$w' = -w_o \tag{9}$$

where the negative sign indicates that the suction velocity is towards the plate.

In view of (9), (8) and (4) now reduce to

$$\frac{\partial u'}{\partial t'} - w_o\frac{\partial u'}{\partial z'} = \frac{\partial U'}{\partial t'} + g_x\beta(T' - T_\infty') + \nu\frac{\partial^2 u'}{\partial z'^2} - \frac{\nu}{\kappa}(u' - U') \tag{10}$$

$$\frac{\partial T'}{\partial t'} - w_o\frac{\partial T'}{\partial z'} = \frac{K}{\rho' C_p}\frac{\partial^2 T'}{\partial z'^2} + \frac{\nu}{C_p}\left(\frac{\partial u'}{\partial z'}\right)^2 - \frac{\nu}{\kappa C_P}u'^2 \tag{11}$$

Introducing the following non-dimensional quantities

$$z = \frac{z' w_o}{\nu},\quad t = \frac{t' w_o^2}{4\nu},\quad u = \frac{u'}{U_o},$$

$$U = \frac{U'}{U_o},\quad \theta = \frac{T' - T_\infty'}{T_w' - T_\infty'},\quad \sigma = \frac{\nu}{w_o\sqrt{\kappa}},$$

$$P = \frac{\mu C_P}{K},\quad G = \frac{\nu g_x\beta(T_w' - T)_\infty'}{U_o w_o^2},$$

$$E = \frac{U_o^2}{C_P(T_w' - T_\infty')} \tag{12}$$

where U_o is the mean of $U'(t)$,
(10) and (11) become

$$\frac{1}{4}\frac{\partial u}{\partial t}-\frac{\partial u}{\partial z}=\frac{1}{4}\frac{\partial U}{\partial t}+G\theta+\frac{\partial^2 u}{\partial z^2}-\sigma^2(u-U) \tag{13}$$

$$\frac{P}{4}\frac{\partial\theta}{\partial t}-P\frac{\partial\theta}{\partial z}=\frac{\partial^2\theta}{\partial z^2}+PE(\frac{\partial u}{\partial z})^2 \tag{14}$$

The corresponding boundary conditions are

$$u=0,\ \theta=1 \text{ at } z=0$$

$$u=U(t),\ \theta=0 \text{ as } z\to\infty \tag{15}$$

For temperature θ to be zero far away from the plate, we drop the term $PE\sigma^2u^2$. Otherwise the temperature θ becomes infinite as $z\to\infty$ which is physically impossible. In the neighbourhood of the plate, we now assume

$$u(z,t)=u_o(z)+\varepsilon e^{i\alpha t}u_1(z),$$

$$\theta(z,t)=\theta_o(z)+\varepsilon e^{i\alpha t}\theta_1(z)$$

and for the stream,

$$u(z,t)=1+\varepsilon e^{i\alpha t} \tag{16}$$

where ε is a small constant quantity ≤ 1 and $\alpha=\frac{4\nu\alpha'}{w_o^2}$ (dimensionless frequency) wherein α' is the frequency of the free stream oscillations. Substituting (16) in (13) and (14) and equating the harmonic terms, neglecting the coefficients of ε^2, we get

$$u_o''+u_o'-\sigma^2(u_o-1)=-G\theta_o \tag{17}$$

$$u_1''+u_1'-\sigma^2(u_1-1)-\frac{i\alpha}{4}u_1=-\frac{i\alpha}{4}-G\theta_1 \tag{18}$$

$$\theta_o''+P\theta_o'=-PEu_o'^2 \tag{19}$$

$$\theta_1''+P\theta_1'-\frac{i\alpha}{4}P\theta_1=-2PEu_o'u_1' \tag{20}$$

where the primes denote differentiation with respect to z. The corresponding boundary conditions are

$$u_o=0,\ u_1=0,\ \theta_O=1,\ \theta_1=0 \text{ at } z=0$$

$$u_o=1,\ u_1=1,\ \theta_o=0,\ \theta_1=0 \text{ as } z\to\infty \tag{21}$$

To solve the coupled non-linear equations (17) to (20), we now assume that the heat due to viscous dissipation is superimposed on the motion. Mathematically this can be achieved by expanding the velocity and temperature terms in powers of E. In the case of incompressible fluids, E is always very small. We now assume

$$u_o(z)=u_{o1}(z)+Eu_{o2}(z)+O(E^2),$$

$$u_1(z)=u_{11}(z)+Eu_{12}(z)+O(E^2)$$

$$\theta_o(z)=\theta_{o1}(z)+E\theta_{o2}(z)+O(E^2),$$

$$\theta_1(z)=\theta_{11}(z)+E\theta_{12}(z)+O(E^2) \tag{22}$$

Substituting (22) in (17) to (21), equating to zero the coefficients of different powers of E and neglecting the terms of $O(E^2)$, we obtain the following set of equations:

$$u_{o1}''+u_{o1}'-\sigma^2(u_{o1}-1)=-G\theta_{o1} \tag{23}$$

$$u_{o2}''+u_{o2}'-\sigma^2u_{o2}=-G\theta_{o2} \tag{24}$$

$$u_{o1}=0,\quad u_{o2}=0 \text{ at } z=0$$

$$u_{o1}=1,\quad u_{o2}=0 \text{ as } z\to\infty \tag{25}$$

$$\theta_{o1}''+P\theta_{o1}'=0 \tag{26}$$

$$\theta_{o2}''+P\theta_{o2}'=-Pu_{o1}'^2 \tag{27}$$

$$\theta_{o1}=1,\ \theta_{o2}=0 \text{ at } z=0$$

$$\theta_{o1}=0,\quad \theta_{o2}=0 \text{ as } z\to\infty \tag{28}$$

$$u_{11}''+u_{11}'-\sigma^2(u_{11}-1)-\frac{i\alpha}{4}u_{11}=-\frac{i\alpha}{4}-G\theta_{11} \tag{29}$$

$$u_{12}''+u_{12}'-(\sigma^2+\frac{i\alpha}{4})u_{12}=-G\theta_{12} \tag{30}$$

$$u_{11}=0,\ u_{12}=0 \text{ at } z=0$$

$$u_{11}=1,\quad u_{12}=0 \text{ as } z\to\infty \tag{31}$$

$$\theta_{11}'' + P\theta_{11}' - \frac{i\alpha}{4}P\theta_{11} = 0 \quad (32)$$

$$\theta_{12}'' + P\theta_{12}' - \frac{i\alpha}{4}P\theta_{12} = -2Pu_{01}'u_{11}' \quad (33)$$

$$\theta_{11} = 0, \quad \theta_{12} = 0 \text{ at } z = 0$$

$$\theta_{11} = 0, \quad \theta_{12} = 0 \text{ as } z \to \infty \quad (34)$$

In this Part I, only the mean flow is described. The equations (23) and (24), (26) and (27) have been solved using the boundary conditions (25) and (28) and discussed the mean flow only in this Part I. The remaining equations for the unsteady flow are discussed and are solved in Part II. The solutions for the mean flow are

$$u_0(z) = 1 + (a_1 - 1)e^{-\lambda z} - a_1 e^{-Pz} + EPGf(z) \quad (35)$$

where

$$a_1 = \frac{G}{P^2 - P - \sigma^2}, \quad \lambda = \frac{1+\sqrt{1+4\sigma^2}}{2},$$

$$f(z) = \left\{a_2(1-a_1)^2 + a_3(1-a_1) + \frac{a_1^2}{2}\right\}$$

$$\times \frac{a_1(e^{-\lambda z} - e^{-Pz})}{G} - a_2(1-a_1)^2$$

$$\times \frac{e^{-\lambda z} - e^{-2\lambda z}}{a_4} - a_3(1-a_1)$$

$$\times \frac{e^{-\lambda z} - e^{-(P+\lambda)z}}{a_5} - \frac{a_1^2(e^{-\lambda z} - e^{-2Pz})}{2a_6}$$

wherein

$$a_2 = \frac{\lambda}{4\lambda - 2P}, \quad a_3 = \frac{2Pa_1}{P+\lambda},$$

$$a_4 = 4\lambda^2 - 2\lambda - \sigma^2$$

$$a_5 = (P+\lambda)^2 - (P+\lambda) - \sigma^2,$$

$$a_6 = 4P^2 - 2P - \sigma^2$$

$$\theta_0(z) = e^{-Pz} + EPg(z) \quad (36)$$

where

$$g(z) = a_2(1-a_1)^2(e^{-Pz} - e^{-2\lambda z})$$

$$+ a_3(1-a_1)\left[e^{-Pz} - e^{-(P+\lambda)z}\right]$$

$$+ \frac{a_1^2}{2}(e^{-Pz} - e^{-2Pz})$$

The mean velocity is shown in Figures 1 to 4 and the mean temperature is shown in Figures 5 and 6, for different values of G, E, P and σ. In order to be more realistic, the values of the Prandtl number are chosen as 0.71 and 7 approximately, which represent air and water respectively at 20°C. The other values of P are chosen arbitrarily. The mean shear stress is obtained as follows:

$$\tau = \frac{\tau'}{U_0 W_0} = \left.\frac{du_0}{dz}\right|_{z=0} \quad (37)$$

Hence from (35) and (37), we get

$$\tau = \lambda + a_1(P - \lambda) + EPGf'(0) \quad (38)$$

where

$$f'(0) = \left\{a_2(1-a_1)^2 + a_3(1-a_1) + \frac{a_1^2}{2}\right\}$$

$$\times \frac{a_1(P-\lambda)}{G} - \lambda a_2 \frac{(1-a_1)^2}{a_4}$$

$$- \frac{a_3 P(1-a_1)}{a_5} - \frac{(2P-\lambda)a_1^2}{2a_6}$$

3. DISCUSSION

The mean velocity for air in the presence of cooled (G > 0) and heated (G < 0) plates is represented in Figs.1 and 2. From Fig. 1, we find that for any value of σ when E is constant, the mean velocity for air increases due to more cooling of the plate by the free convection currents. From Table 1, we see that when G is doubled, for σ = 0 there is an increase of 115% in the maximum velocity, whereas for σ = 10 the increase in the maximum velocity is only 2.5%. Similarly, when the value of E is doubled, there is an increase of 5% in the maximum velocity for σ = 0, while for σ =10 the increment is only 0.01%. From Fig. 2, we observe that near the boundary layer the mean velocity for air in the presence of heated plate is negative for σ = 0. As σ increases, this gradually decreases in magnitude and becomes positive with further increase in σ. Away from the boundary layer, the mean velocity increases as σ increases. From Table 2, we observe that when the value of G is doubled, there is 66% decrease in the maximum velocity for σ = 0 whereas for σ = 10 the decrement is only 2.5% and when the value of E is doubled, the increase in the maximum velocity varies from 10% to 0.002% as σ varies from 0 to 10.

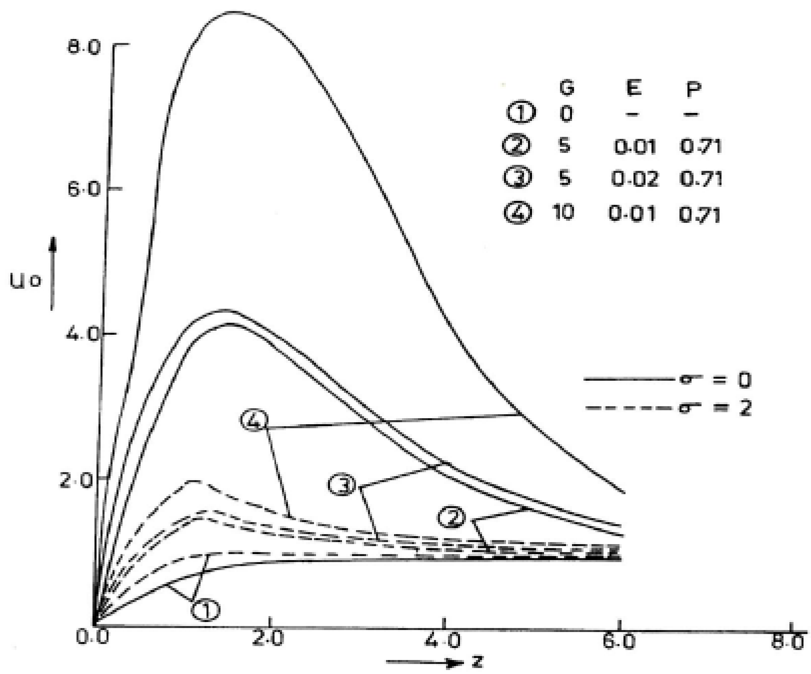

Fig. 1 Mean velocity for air when G ≥ 0

Table 1 - Percentage increase in the maximum velocity when the plate is cooled for air

σ	When G is doubled (increase)	When E is doubled (increase)
0	115.0	5.00
5	8.9	0.03
10	2.5	0.01

Table 2 - Percentage increase (or decrease) in the maximum velocity when the plate is heated for air

σ	When G is doubled (decrease)	When E is doubled (increase)
0	66.0	10.000
5	11.0	0.020
10	2.5	0.002

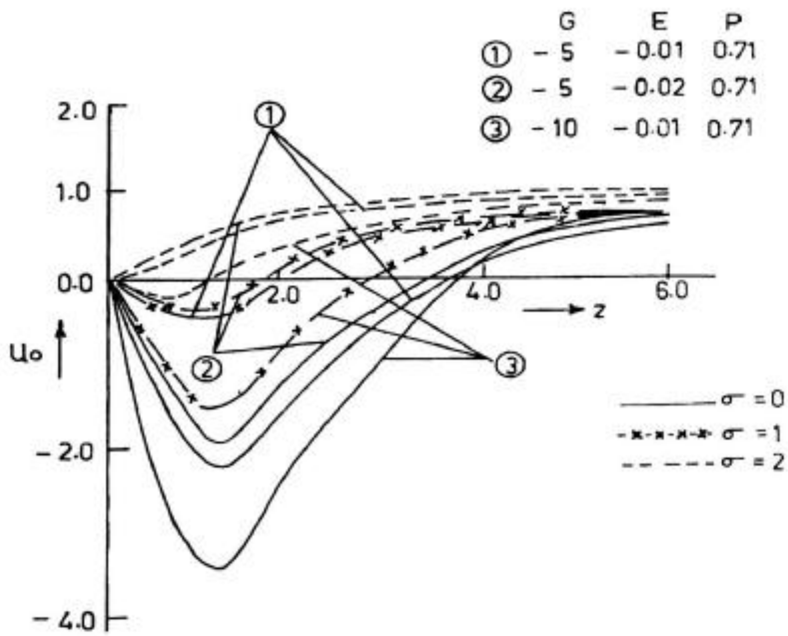

Fig. 2 Mean velocity for air when G < 0

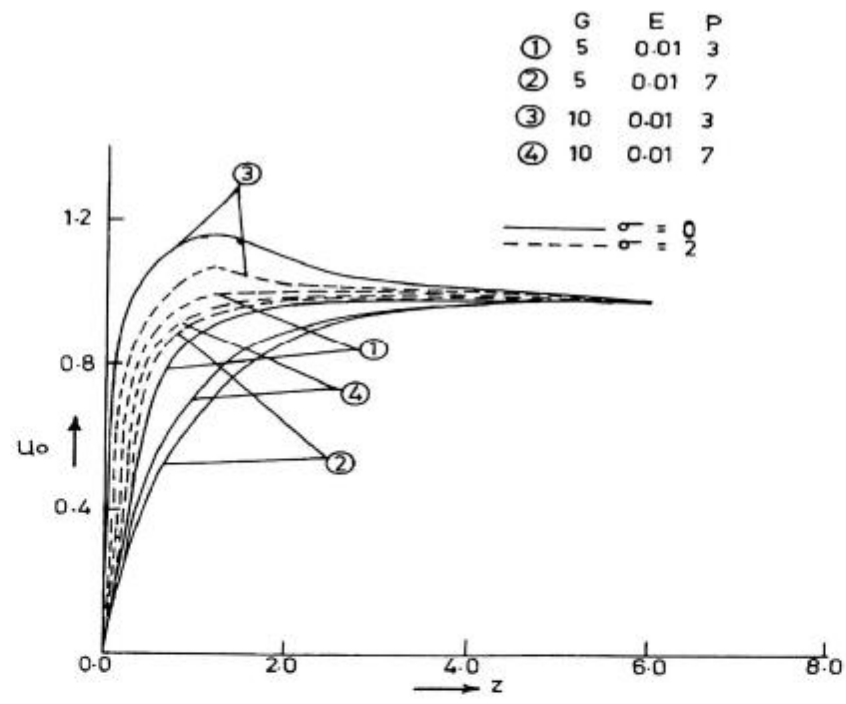

Fig. 3(a) Mean velocity for fluids with increasing Prandtl number when G > 0

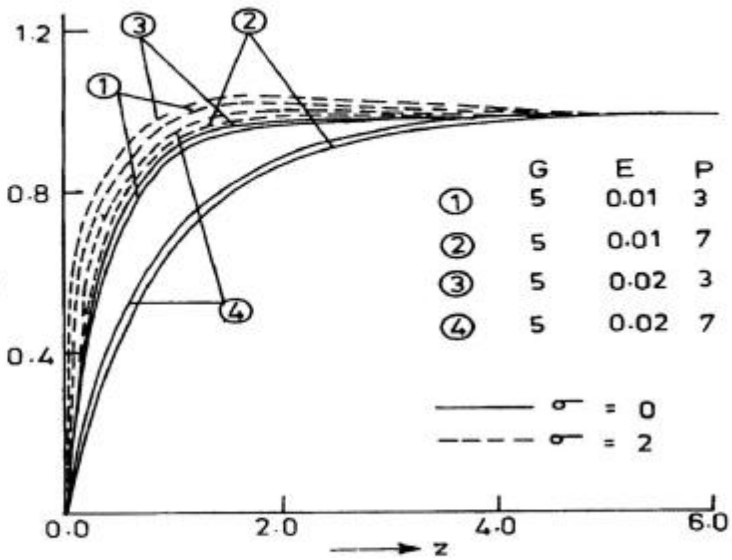

Fig. 3(b) Mean velocity for fluids with increasing Prandtl number when G > 0 with more addition of heat

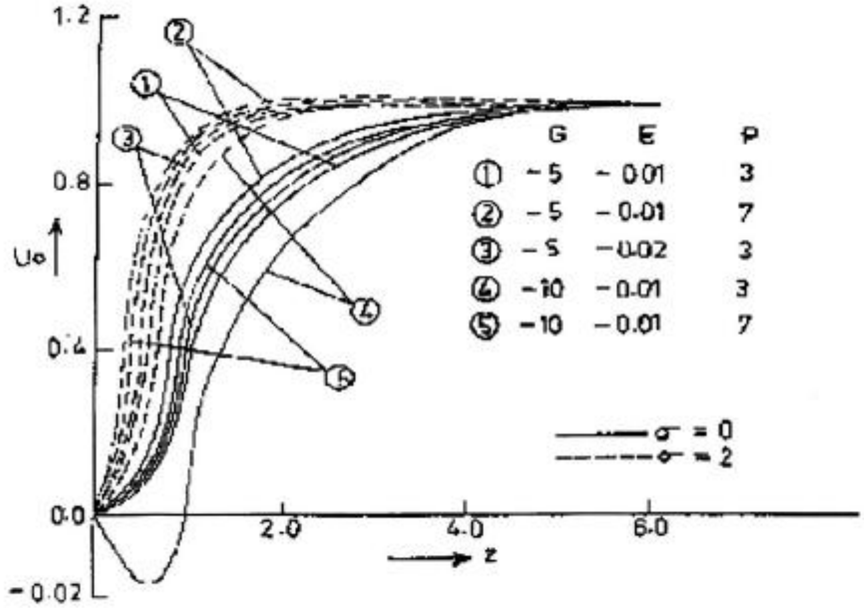

Fig. 4 Mean velocity for fluids with increasing Prandtl number when G < 0

The mean velocity for fluids with increasing Prandtl numbers in the case of cooling and heating of the plate for different values of σ is represented in Figs. 3 and 4 respectively. From Fig. 3(a), we find that the mean velocity profiles are positive in the case of cooling of the plate. With more cooling of the plate (1,3,2,4) the mean velocity increases. From Fig. 3(b), we observe that with more addition of heat due to viscous dissipation, the mean velocity also increases (1,3,2,4).

From Fig. 4, we observe that the mean velocity profile is of separated type in the case of fluids with small Prandtl number, namely, $P = 3$. For fluids with large Prandtl numbers, for any value of σ, the mean velocity decreases (2,5) with more heating of the plate. But when the value of G doubled, the percentage decrease in the maximum velocity decreases as σ increases. When the value of E is doubled there is a percentage increase in the maximum velocity, which decreases as σ increases from 0 to 10. The mean velocity increases with increasing Prandtl numbers (1,2 and 4,5) in case of heating of the plate.

From numerical calculations, it is found that with addition of heat due to viscous dissipation, for any value of σ the mean temperature for air increases irrespective of the absence or the presence of free convection currents. Due to greater cooling of the plate, for air, the mean temperature in the boundary layer increases with increasing σ. For air, the effect of increasing σ is to decrease the mean temperature when the plate is cooled.

The mean temperature in the case of liquids with increasing Prandtl numbers for different values of σ is shown in Fig. 5. From this figure, we find that the mean temperature decreases as the Prandtl number increases. For any value of σ, the mean temperature of water at $20^{0}C$ ($P = 7$) decreases owing to greater cooling of the plate. For fluids with low Prandtl numbers and $\sigma = 0$, the decrease in temperature is more significant than that with high Prandtl numbers and $\sigma > 0$. For fluids, the effect of increasing σ is to increase the mean temperature when the plate is cooled.

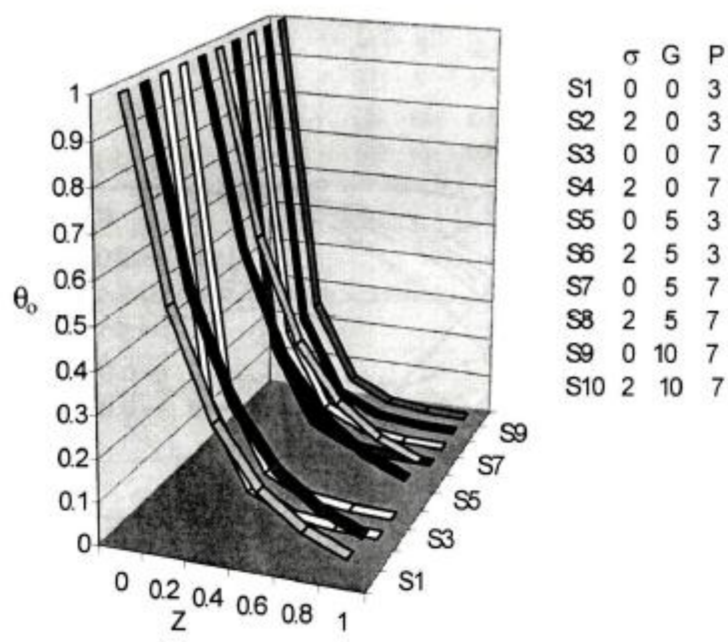

Fig. 5 Mean temperature for fluids when $G \geq 0$ and $E = 0.01$

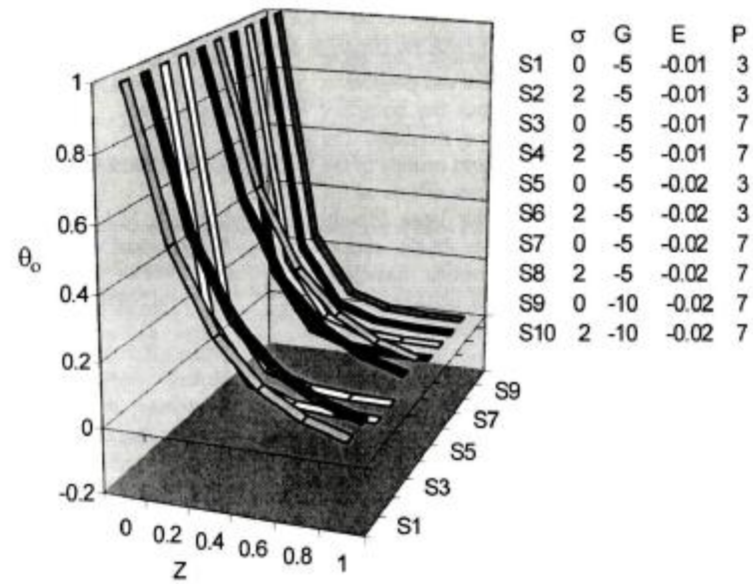

Fig. 6 Mean temperature for fluids when $G < 0$

The mean temperature in the case of heating of the plate for different values of σ is shown in Fig. 6. From this figure, we observe that in the case of water the mean temperature decreases, owing to greater heating of the plate. For fluids, with more addition of heat due to viscous dissipation, the mean temperature decreases. In the case of heating of the plate the mean temperature decreases with increasing Prandtl numbers for any value of σ.

From the numerical calculations, it is found that for any value of σ, for both air and liquids, there is an increase in the mean shearing stress owing to greater cooling of the plate, whereas owing to greater heating of the plate there is a decrease in the mean shearing stress.

The percentage increase (or decrease) in the mean shearing stress when the plate is cooled/heated is shown in Table 3/Table 4. From these tables, we observe that when the value of G is doubled (with greater cooling of the plate), the effect of increasing σ is to decrease the mean shearing stress from 117% and 41% to 4.5% and 3% in case of air and water respectively. In the presence of heated plate, when the value of G is doubled, the effect of increasing σ is to decrease the mean shearing stress from 72% and 230% to 16% and 3% in case of air and water respectively. With an increase in the Prandtl number the mean shearing stress, for any value of σ, decreases for $G > 0$ and increases for $G < 0$. Also with more addition of heat due to viscous dissipation, the mean shearing stress increases with increasing σ, in case of both cooling and heating of the plate.

Table 3 - Percentage increase in the mean shearing stress when the plate is cooled

σ	When G is doubled	
	for air	for water
0	117.0	41.0
5	15.0	7.5
10	4.5	3.0

Table 4 - Percentage decrease in the mean shearing stress when the plate is heated

σ	When G is doubled	
	for air	for water
0	72	230
5	21	9
10	16	3

4. CONCLUSIONS

Near the boundary layer, the mean velocity for air is negative in the presence of heated plate and positive in the case of cooled plate. Away from the boundary layer, the mean velocity increases as σ increases. The mean velocity for air increases with more cooling of the plate. Heating and cooling have separate effects on the mean velocity profiles of fluids with large Prandtl numbers irrespective of the value of σ. In the case of heating of the plate the mean velocity increases, whereas in the case of cooling of the plate the mean velocity decreases with increasing Prandtl numbers. The percentage increase (or decrease) in the maximum velocity for air is considerably more in the absence of σ and less in the presence of σ when the plate is cooled or heated. It decreases as σ increases.

The mean temperature for air increases with addition of heat due to viscous dissipation irrespective of the presence or absence of the free convection currents and it decreases as the Prandtl number increases. For fluids, with more addition of heat, the mean temperature decreases.

Irrespective of the value of σ, the mean shearing stress increases owing to greater cooling of the plate whereas it decreases owing to greater heating of the plate. With more addition of heat, the mean shearing stress increases with increasing σ in the case of both cooling and heating of the plate.

5. ACKNOWLEDGEMENTS

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