

Lie group classification of the generalized Lane–Emden equation

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A B S T R A C T

We carry out the Lie group classification of the generalized Lane–Emden equation $xu'' + nu' + xH(u) = 0$, which has many applications in mathematical physics and astrophysics. We show that the equation admits a three-dimensional equivalence Lie algebra. It is also shown that the principal Lie algebra, which in this case is trivial, has seven possible extensions. Three new cases arise for which the Lie point symmetry algebra is non-trivial. Comparison is then made of these cases with the Noether symmetry cases as well as the partial Noether operators.

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Noether point symmetries

Partial Noether operators

1. Introduction

The Lane–Emden equation [1,2]

$$\frac{d^2u}{dx^2} + \frac{2}{x} \frac{du}{dx} + u^r = 0, \quad (1)$$

where r is a constant, models the equilibria of non-rotating fluids in which internal pressure balances self-gravity. When spherically symmetric solutions of Eq. (1) appeared in [3], they got the attention of astrophysicists. In the latter half of the twentieth century, some interesting applications of the isothermal solution (singular isothermal sphere) and its non-singular modifications were used in the structures of collisionless systems such as globular clusters and early-type galaxies [4,5].

The work of Emden [2] also got the attention of physicists outside the field of astrophysics [6–8] who investigated the generalized polytropic forms of the Lane–Emden equation (1) for specific polytropic indices r . Some singular solutions for $r = 3$ were produced by Fowler [6,9] and the Emden–Fowler equation in the literature was established, while the works of Thomas [10] and Fermi [8] resulted in the Thomas–Fermi equation, used in atomic theory. Both of these equations, even today, are being investigated by physicists and mathematicians.

Many problems in mathematical physics and astrophysics can be formulated by the generalized Lane–Emden equation

$$\frac{d^2u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(u) = 0, \quad (2)$$

where n is a real constant. For $n = 2$ the approximate analytical solutions to Eq. (2) were studied by Wazwaz [11] and Dehghan and Shakeri [12].

Recently, Khalique et al. [13] investigated the Noether point symmetries of the generalized Lane–Emden equation (2) and reduced the corresponding Lane–Emden equations to quadratures which admitted Noether point symmetries.

Many applications of the Lane–Emden equation and several methods for its solution have been given in the literature. The interested reader is referred to the references given in [13].

The purpose of this paper is to study the Lie group classification of the generalized Lane–Emden equation (2). We also compare our results to the Noether and partial Noether cases.

The problem of group classification was first carried out by Lie [14] in 1881. Later, various people including Ovsiannikov applied Lie's methods to a wide range of physically important problems. The problem of group classification of a differential equation involving an arbitrary element, say H , consists of finding the Lie point symmetries of the differential equation with arbitrary H , and then determining all possible forms of H for which the symmetry group can be extended. The reader is referred to a recent excellent paper [15] on this aspect.

For the applications of symmetry analysis (Lie group analysis) to differential equations see, for example [16–19].

The rest of this paper is organized in the following way: In Section 2, we calculate the equivalence transformations of Eq. (2). We determine the principal Lie algebra and perform the complete Lie group classification of Eq. (2) in Section 3. Then in Section 4, we compare the Lie and Noether symmetries and in the subsequent Section 5 partial Noether operators are used to find first integrals for three cases. Finally, concluding remarks are made in Section 6.

2. Equivalence transformations

We recall that [20] an equivalence transformation of Eq. (2) is an invertible transformation of the variables x and u mapping Eq. (2) into an equation of the same form, where the form of the transformed function can, in general, be different from the form of the original function $H(u)$. We now write Eq. (2) as the system

$$\frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(u) = 0, \quad H_x = 0, \quad (3)$$

where u is a differential variable with independent variable x and H is a differential variable with independent variables x and u and calculate the generators of the continuous group of equivalence transformations in the form

$$Y = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + \mu(x, u, H) \frac{\partial}{\partial H}. \quad (4)$$

We apply the Lie's infinitesimal technique by using the prolongation of Y to the derivatives involved in the system (3) as follows (see, e.g., [16]):

$$\tilde{Y} = Y + \zeta_x \frac{\partial}{\partial u'} + \zeta_{xx} \frac{\partial}{\partial u''} + \omega_x \frac{\partial}{\partial H_x},$$

where ζ_x and ζ_{xx} are given by the usual prolongation formulas and the operator ω_x is determined by

$$\omega_x = \tilde{D}_x(\mu) - H_x \tilde{D}_x(\xi) - H_u \tilde{D}_x(\eta).$$

Here

$$\tilde{D}_x = \frac{\partial}{\partial x} + H_x \frac{\partial}{\partial H} + \dots$$

is the new total differentiation for the system (3).

The use of invariance test for equations (3) requires that the following system of determining equations holds:

$$\tilde{Y} \left(\frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H(u) \right) \Big|_{(3)} = 0, \quad \tilde{Y}(H_x) \Big|_{(3)} = 0. \quad (5)$$

Solving the above equations we conclude that the system (3) has the three-dimensional equivalence Lie algebra spanned by the equivalence generators

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial u}, \\ Y_2 &= x \frac{\partial}{\partial x} - 2H \frac{\partial}{\partial H}, \\ Y_3 &= u \frac{\partial}{\partial u} + H \frac{\partial}{\partial H}. \end{aligned} \quad (6)$$

and hence the three-parameter equivalence group is given by

$$\begin{aligned} \bar{x} &= e^{a_2} x, \\ \bar{u} &= e^{a_3} u + a_1, \\ \bar{H} &= e^{a_3 - 2a_2} H. \end{aligned} \quad (7)$$

3. Principal Lie algebra and Lie group classification

The generalized Lane–Emden equation (2) admits a Lie point symmetry

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}$$

if

$$X^{[2]} \left(\frac{d^2 u}{dx^2} + \frac{n}{x} \frac{du}{dx} + H \right) \Big|_{(2)} = 0. \quad (8)$$

After straightforward and lengthy calculations, the above determining equation yields

$$\begin{aligned} \xi &= b(x), \\ \eta &= c(x)u + d(x), \\ -\frac{n}{x^2}b + \frac{n}{x}b' + 2c' - b'' &= 0 \end{aligned} \quad (9)$$

and

$$(cu + d)H'(u) + (2b' - c)H(u) + \left(\frac{n}{x}c' + c''\right)u + \left(\frac{n}{x}d' + d''\right) = 0. \quad (10)$$

Consequently, we conclude that the principal Lie algebra of (2) is trivial and our classifying relation is

$$(\alpha u + \beta)H' + \lambda H + \gamma u + \delta = 0, \quad (11)$$

where $\alpha, \beta, \lambda, \gamma$ and δ are constants. This classifying relation is invariant under the equivalence transformations (7) if

$$\bar{\alpha} = \alpha, \quad \bar{\beta} = (\alpha a_1 + \beta)e^{-a_3}, \quad \bar{\lambda} = \lambda, \quad \bar{\gamma} = \gamma e^{2a_2}, \quad \bar{\delta} = e^{2a_2 - a_3}(\gamma a_1 + \delta). \quad (12)$$

The above relations are now used to find the non-equivalent forms of H . This prompts the following eight cases.

Case 1. $n \neq 0$, $H(u)$ is arbitrary, but not of the form contained in Cases 3–6.

There is no Lie point symmetry in this case.

Case 2. $n = 0$, $H(u)$ is arbitrary, but not of the form contained in Cases 4–6.

In this case, for the corresponding Eq. (2), we obtain a single Lie point symmetry

$$X = \frac{\partial}{\partial x}. \quad (13)$$

Case 3. $H(u)$ is linear in u .

The corresponding Eq. (2) in this case has $sl(3, \mathfrak{R})$ symmetry algebra and is well known in the literature (see, for example, [21, 22]).

Case 4. $H(u) = C - \delta u^2/2$, where C is a constant and $\delta = \pm 1$.

Here five subcases arise:

4.1. $n = 5$, $C = 0$. The corresponding Eq. (2) admits just one Lie point symmetry, namely,

$$X_1 = x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u}. \quad (14)$$

4.2. $n = 25$, $C = 0$. In this subcase, the same Lie point symmetry as in Case 4.1 is admitted by the corresponding Eq. (2).

4.3. $n = 5/3$, $C = 0$. The corresponding Eq. (2) admits two Lie point symmetries. One given by the operator (14) and the second by

$$X_2 = x^{1/3} \frac{\partial}{\partial x} - \frac{2}{3} x^{-2/3} u \frac{\partial}{\partial u}.$$

4.4. $n = 15$, $C = 0$. The corresponding Eq. (2) admits a two-dimensional symmetry Lie algebra, which is spanned by the symmetry operators (14) and

$$X_2 = x^3 \frac{\partial}{\partial x} - \left(6x^2 u + \frac{192}{\delta} \right) \frac{\partial}{\partial u}.$$

4.5. $n = 10/3$, $C = 0$. In this subcase the corresponding Eq. (2) admits two Lie point symmetries, one given by (14) and the second by

$$X_2 = x^{2/3} \frac{\partial}{\partial x} - \left(\frac{4}{3} x^{-1/3} u - \frac{8}{9\delta} x^{-7/3} \right) \frac{\partial}{\partial u}.$$

4.6. $n = 0$, $C = 0$. The corresponding Eq. (2) admits a two-dimensional symmetry Lie algebra, which is spanned by the operators (13) and (14).

Case 5. $H(u) = -\delta_1/v - u \delta_2/(v+1) + Cu^{-v}$, where $\delta_1, \delta_2 = 0, \pm 1, v \neq -1, 0$ and C is a constant.

Here three subcases arise:

5.1. $n = \frac{v-3}{v+1}, v \neq 3, \delta_1, \delta_2 = 0$. A single Lie point symmetry generator

$$X_1 = x \frac{\partial}{\partial x} + \frac{2}{v+1} u \frac{\partial}{\partial u} \quad (15)$$

is admitted by the corresponding Eq. (2).

5.2. $n = \frac{v-3}{v-1}, v \neq 3, \delta_1, \delta_2 = 0$. The corresponding Eq. (2) admits two Lie point symmetries, namely, (15) and

$$X_2 = x^{\frac{v+1}{v-1}} \frac{\partial}{\partial x} + \frac{2}{v-1} x^{\frac{2}{v-1}} u \frac{\partial}{\partial u}. \quad (16)$$

5.3. $n = 0, \delta_1, \delta_2 = 0$. This corresponds to $v = 3$ in Cases 5.1 and 5.2. Here the corresponding Eq. (2) admits a three-dimensional symmetry Lie algebra spanned by the operators (15) and (16) with $v = 3$ and (13).

Case 6. $H(u) = Ce^{-\delta_1 u} + \delta_2 u + \delta_3$, where $\delta_1 = \pm 1, \delta_2, \delta_3 = 0, \pm 1$ and C is a constant.

Three subcases arise. These are

6.1. For all values of $n \neq 0, 1, \delta_2, \delta_3 = 0$, the corresponding Eq. (2) admits a single Lie point symmetry

$$X_1 = x \frac{\partial}{\partial x} + \frac{2}{\delta_1} \frac{\partial}{\partial u}. \quad (17)$$

6.2. $n = 1, \delta_2, \delta_3 = 0$. In this subcase the corresponding Eq. (2) admits two Lie point symmetries. The symmetry operator (17) and

$$X_2 = x \ln x \frac{\partial}{\partial x} + \frac{2}{\delta_1} (1 + \ln x) \frac{\partial}{\partial u}.$$

6.3. $n = 0, \delta_2, \delta_3 = 0$. The corresponding Eq. (2) admits two Lie point symmetries and these are (13) and (17).

Case 7. $H(u) = -\delta_1 \ln u - \delta_2 u + C$, where $\delta_1, \delta_2 = 0, \pm 1$ and C is a constant.

This reduces to case 2.

Case 8. $H(u) = -\delta_1 u \ln u + Cu + \delta_2$, where $\delta_1, \delta_2 = 0, \pm 1$ and C is a constant.

This reduces to case 2.

4. Comparison of Lie and Noether symmetries

Here we compare the Lie point symmetries of the generalized Lane–Emden equation (2) for various functions $H(u)$ obtained in Section 3 with the Noether point symmetries given in [13]. In Case 1 of Section 3 we observe that Eq. (2) has no Lie point symmetry. No Noether point symmetry exists in this case also. When $n = 0$ and $H(u)$ arbitrary, the corresponding Eq. (2) has one Noether point symmetry and one Lie point symmetry, except for the Cases 4–6 where we have two, three and two Lie point symmetries, respectively.

In the case when the function $H(u)$ is linear, it is well known in the literature that the corresponding Eq. (2) has $sl(3, \mathbb{R})$ symmetry algebra and five Noether point symmetries associated with the standard Lagrangian of (2) (see, e.g., [21,22]).

When $H(u)$ is quadratic in u , we get one Noether point symmetry (14) for every case, but either one or two Lie point symmetries.

For power function, viz., $H(u) = Cu^{-v}$ and $n = \frac{v-3}{v+1}$ we get a single Lie point symmetry and a single Noether point symmetry, whereas, when $n = \frac{v-3}{v-1}$, we obtain two Lie point symmetries but one Noether point symmetry.

The exponential case, i.e., when $H(u) = Ce^{-\delta_1 u}$ and $n \neq 1$, yields no Noether point symmetry but one Lie point symmetry. But for $n = 1$ we obtain one Noether and two Lie point symmetries.

Finally, in the last two cases when $H(u)$ contains $\ln u$ term, we obtain single Noether and single Lie point symmetry.

5. First integrals via partial Noether operators

Here we employ the recently introduced notion of partial Noether method [23] (see also [24]) to construct first integrals for the generalized Lane–Emden equation (2) for some functions $H(u)$ obtained in Section 3. The reader is referred to [23,24] for the details of definitions and theorems concerning these new concepts. We consider first Case 5.1. viz., the generalized Lane–Emden equation

$$\frac{d^2 u}{dx^2} + \frac{v-3}{v+1} \frac{1}{x} \frac{du}{dx} + Cu^{-v} = 0, \quad v \neq -1, 0, 3. \quad (18)$$

In [13], Noether symmetries and first integral for Eq. (18) have been investigated. Here, we employ the partial Noether method to find the first integral. Using the partial Lagrangian $L = u^2/2$ we have

$$\frac{\delta L}{\delta u} = \frac{v-3}{v+1} \frac{u'}{x} + Cu^{-v}$$

and the determining equation gives

$$\xi = C_1 x^{\frac{2(v-1)}{v+1}}, \quad \eta = \frac{2C_1}{v+1} u x^{\frac{v-3}{v+1}}, \quad B = \frac{C_1 C}{1-v} u^{1-v} x^{\frac{2(v-1)}{v+1}}.$$

Consequently the partial Noether operator is

$$X = x^{\frac{2(v-1)}{v+1}} \frac{\partial}{\partial x} + \frac{2}{v+1} u x^{\frac{v-3}{v+1}} \frac{\partial}{\partial u}.$$

(Note that this is not a Lie point symmetry of Eq. (18).) Invoking Theorem 1 in [24], the first integral of equation (18) is given by

$$I = \frac{C}{1-v} x^{\frac{2(v-1)}{v+1}} u^{1-v} + \frac{1}{2} x^{\frac{2(v-1)}{v+1}} u'^2 - \frac{2}{1+v} x^{\frac{v-3}{v+1}} u u',$$

which happens to be the same as found in [13].

For Case 5.2., the generalized Lane–Emden equation is

$$\frac{d^2 u}{dx^2} + \frac{v-3}{v-1} \frac{1}{x} \frac{du}{dx} + Cu^{-v} = 0, \quad v \neq -1, 0, 3. \quad (19)$$

Using the partial Lagrangian $L = u^2/2$ and employing the method given in [23] the partial Noether operator is

$$X = x^2 \frac{\partial}{\partial x} + \frac{2}{v-1} x u \frac{\partial}{\partial u}.$$

and the application of Theorem 1 in [24] yields the first integral

$$I = \frac{C}{1-v} x^2 u^{1-v} + \frac{1}{2} x^2 u'^2 + \frac{2}{1-v} x u u' + \frac{2}{(1-v)^2} u^2.$$

This is same as found in [13].

Finally, we consider the exponential case, i.e., $H(u) = Ce^{-\delta_1 u}$ with $n = 1$. In this case the corresponding Lane–Emden equation is

$$\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + Ce^{-\delta_1 u} = 0. \quad (20)$$

Taking the same partial Lagrangian $L = u^2/2$ yields the partial Noether operator

$$X = x^2 \frac{\partial}{\partial x} + \frac{2}{\delta_1} x \frac{\partial}{\partial u}.$$

and the first integral

$$I = -\frac{C}{\delta_1} x^2 e^{-\delta_1 u} + \frac{1}{2} x^2 u'^2 - \frac{2}{\delta_1} x u',$$

which is the same as found in [13] using the Noether approach.

One can also deduce the first integrals for other cases in a similar way.

6. Concluding remarks

In this paper, we have used Lie group analysis to perform a complete group classification of the generalized Lane–Emden equation (2). We provided a motivation for the algebraic study of (2) in Section 1. We showed that the equation admits a three-dimensional equivalence Lie algebra. The principal Lie algebra, which was found to be trivial, had seven possible extensions. Three new cases were found and these corresponded to Cases 4–6 of Section 3. We then compared these results with the Noether symmetry cases given in [13]. Finally, first integrals of three of these cases were computed using the recently introduced notions of partial Lagrangian and partial Noether operator and then were compared with the Noether approach.

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