A study of pulsatile blood flow modeled as a power law fluid in a constricted tube ☆

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Abstract

A mathematical model for the pulsatile blood flow in a small vessel in the cardiovascular system with a mild stenosis is analyzed. Blood is modeled as a power law fluid and the differential approximation for the heat flux is invoked in the energy equation. The effect of heat transfer on the velocity is computed and discussed.

Keywords: Pulsatile blood flow; Stenosis

1. Introduction

Berger and Jou [1] have shown that the relationship between flow in arteries and the sites where atherosclerosis develops are linked. It is well known that the flow of blood depends on the pumping action of the heart which gives blood flow its pulsatile nature. Chakravarty and Sannigrahi [2] reported a study of an analytical estimate of the flow field in a porous stenotic artery subject to body acceleration where it was observed that when the body acceleration is withdrawn from the system under study the wall shear stresses are enhanced largely through out the constricted arterial segment under consideration. Dash et al. [4] and El-Shaded [5] also studied the effect of body acceleration on the flow field in a porous stenotic artery. Misra and Chauhan [7] reported the results of a study of pulsatile blood flow in a tube with pulsating walls where blood was modeled as a two-layered fluid. Ogulu and Abbey [9] studied the effect of heat transfer during deep heat muscle treatment in their study on oscillatory blood flow in an indented porous artery. Other studies of interest in this area include Yang and Fu [3], Ju et al. [6], Guo et al. [8], Moh et al. [10], Tagawa and Ozoe [12], Fusegi [14] and Sekar and Nath [17].

The studies mentioned above are a few out of the literature on physiological fluid dynamics. The studies mentioned above all treat blood as a Newtonian fluid but as suggested in Majhi and Nair [11], blood behaves like a non-Newtonian fluid under certain conditions. In this study we propose a mathematical model for pulsatile blood flow treating blood as a non-Newtonian power law fluid. It is hoped that this study will compliment the study

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Nomenclature

A_0	Amplitude of the fluctuating component
u	Axial component of the velocity
μ_0	Coefficient of viscosity of blood plasma
A	Constant component of the pressure gradient
Q	Flow rate
G_r	Free convection parameter
f_p	Heart pulse frequency
N,μ	Parameters of the fluid
Pe	Peclett number
p	Pressure
N	Radiation parameter
R_0	Radius of the normal tube
$R(x)$	Radius of the stenosed artery
τ_{rx}	Shear stress
θ	Temperature

reported in Prakash et al. [13] and extend the applicability of this and other studies in this rapidly growing area of physiological fluid dynamics.

2. Mathematical formulation of the problem

We start with the problem as formulated in Prakash et al. [13]. In short we consider steady, laminar and axially symmetrical flow of blood through an artery provided with a mild stenosis, where blood is modeled as a non-Newtonian fluid and the blood vessel is heated externally. We assume the temperature difference between the flowing blood and the blood vessel is high enough for heat transfer to be significant, Ogulu and Abbey [9].

Under the assumption of small electrical conductivity and the usual Boussinesq approximation, the proposed non-dimensional governing equations of the flow are:

$$\frac{\partial p}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rx})+G_r^*\theta=0 \tag{1}$$

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+N^2\theta=0 \tag{2}$$

where p is the pressure, u the axial component of the velocity, θ is the temperature and Pe is Peclett number, $G_r^*=\frac{G_rR_0^2}{\mu_0}$, where G_r is the free convection parameter, R_0 is the radius of the normal tube, μ_0 is the coefficient of viscosity of blood plasma, N is the radiation parameter and $R(x)$ will denote the radius of the stenosed artery. The idealized geometry of the stenosis is given in Fig. 1 above.

The shear stress τ_{rx} is defined as

$$\tau_{rx}=-\mu\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^{n-1}\frac{\mathrm{d}u}{\mathrm{d}r} \tag{3}$$

where n and μ are parameters of the fluid. For $n=1$, Eq. (3) reduces to Newton's law of viscosity with $m=\mu$: thus deviation from unity indicates the degree of deviation from Newtonian behavior. For values of $n<1$, it is pseudo plastic, where as for $n>1$ it is dilatant.

In this model we re-write Eq. (3) as

$$\tau_{rx}=-\mu\left|\frac{\mathrm{d}u}{\mathrm{d}r}\right|^{\gamma}. \tag{4}$$

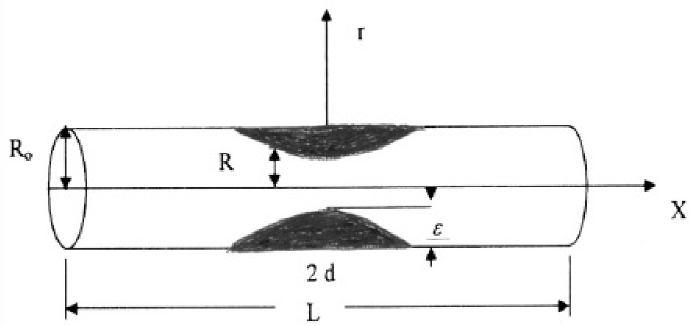

Fig. 1. Geometry of the stenosis.

As in Fitz-Gerald [15] where μ is a function of τ_{rx} and $n=1$. τ_{rx} is the only non-zero shear component, Ogulu and Bawo [16].

In Eq. (2) we have assumed blood is an optically thin fluid, Ogulu and Abbey [9], with a low relative density and $\alpha \ll 1$, Ogulu and Bestman [18], so that in the spirit of Cogley et al. [19], we expressed the heat flux as

$$\frac{\partial q}{\partial r} = 4\alpha^2(\theta - \theta_\infty) \tag{5}$$

where

$$\alpha^2 = \int_0^\infty \gamma\lambda \frac{\partial B}{\partial \theta} \tag{6}$$

Substituting Eq. (4) into Eq. (1) gives

$$-\frac{1}{r}\frac{\partial}{\partial r}\left\{ r\mu \left| \frac{\mathrm{d}u}{\mathrm{d}r} \right|^{\gamma} \right\} = -\frac{\partial p}{\partial x} - G_r\theta. \tag{7}$$

Now, we put

$$-\frac{\partial p}{\partial x} = p_0 e^{i\omega t} \tag{8a}$$

$$\theta(r,t) = \theta_0(r) e^{i\omega t} \tag{8b}$$

$$u(r,t) = u_0(r) e^{i\omega t} \tag{8c}$$

Then, Eq. (7) becomes

$$p_0 = -\frac{1}{r}\frac{\partial}{\partial r}\left\{ r\mu \left| \frac{du_0}{dr} \right|^{\gamma} \right\} + G_r\theta_0. \tag{9}$$

Next we introduce the transformation

$$y = \frac{r}{R_0}$$

into Eq. (9) so that we have

$$-\frac{p_0 R_0^{\gamma+1}}{\mu}+G_r\theta_0\frac{R_0^{\gamma+1}}{\mu}=\frac{1}{y}\frac{\partial}{\partial y}\left\{y\left|\frac{\mathrm{d}u_0}{\mathrm{d}y}\right|^{\gamma}\right\}. \tag{10}$$

The boundary conditions are now given as in Prakash et al. [13],

$$\begin{aligned}&u_0=0,\quad \theta_0=\theta_w\quad \text{on } y=\frac{R(x)}{R_0}\\&u,\theta,<\infty \text{ on } y=0\\&\frac{\mathrm{d}u}{\mathrm{d}y}=0 \text{ on } y=0\end{aligned} \tag{11}$$

The mathematical statement of the problem is now complete.

3. Analysis

The solution of Eq. (2) is straight forward, it is

$$\theta(r,t)=\theta_w\frac{J_0(N\,y)}{J_0\left(N\frac{R(x)}{R_0}\right)} \tag{12}$$

First from Eq. (12) we get,

$$\theta_0(0,t)=\theta_w\frac{J_0(0)}{J_0\left(N\frac{R(x)}{R_0}\right)}=\theta_w\frac{1}{J_0\left(N\frac{R(x)}{R_0}\right)} \tag{13}$$

Putting this into Eq. (10) gives:

$$\frac{1}{y}\frac{\partial}{\partial y}\left\{y\left|\frac{\mathrm{d}u_0}{\mathrm{d}y}\right|^{\gamma}\right\}=\left[G_r^*\theta_w\frac{1}{J_0\left(N\frac{R(x)}{R_0}\right)}-p_0\right]\frac{R_0^{\gamma+1}}{\mu}. \tag{14}$$

On integration subject to the stated boundary conditions in Eq. (10), we get

$$u_o=\left[\left\{\frac{G_r^*\theta_w}{J_0\left(N\frac{R(x)}{R_0}\right)}-p_0\right\}^{\xi-1}\frac{R_0^{\xi}}{(2\mu)^{\xi-1}\xi}\right]\left(y^{\xi}-\left(\frac{R(x)}{R_0}\right)^{\xi}\right) \tag{15}$$

where $\xi=\frac{\gamma+1}{\gamma}$ and $p_0=A+A_0\sin(\omega_p t)$. A is the constant component of the pressure gradient, A_0 is the amplitude of the fluctuating component and $\omega_p=2\pi f_p$ where f_p is the heart pulse frequency.

The flow rate Q is defined as

$$Q=2\pi\int_0^{R_0} y u_0\,\mathrm{d}y \tag{16}$$

On simplification we get,

$$Q=\frac{2\pi R_0^2}{\xi(\xi+2)(2\mu)^{\xi-1}}\left(R_0^{2\xi}-\frac{(\xi+2)}{2}(R(x)^{\xi})\right)\left\{\frac{G_r^*\theta_\omega}{J_0\left(N\cdot\frac{R(x)}{R_0}\right)}-p_0\right\}$$

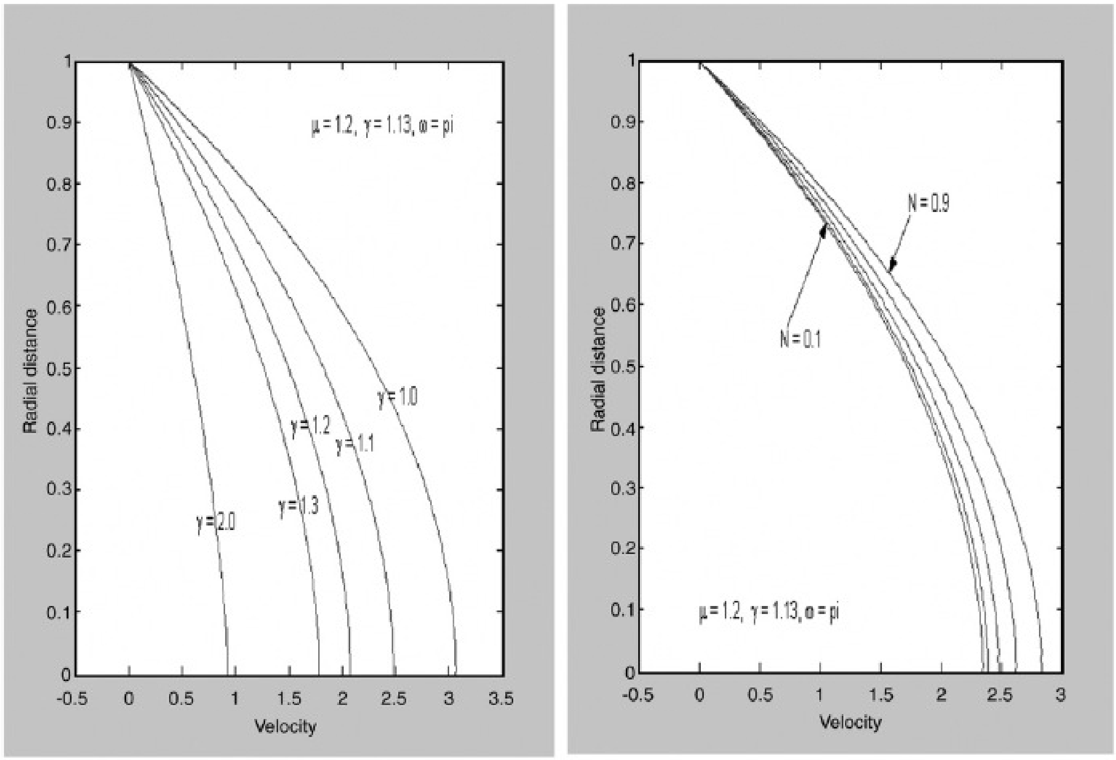

Fig. 2. Velocity profiles showing the effects of radiation, N and exponential index, γ.

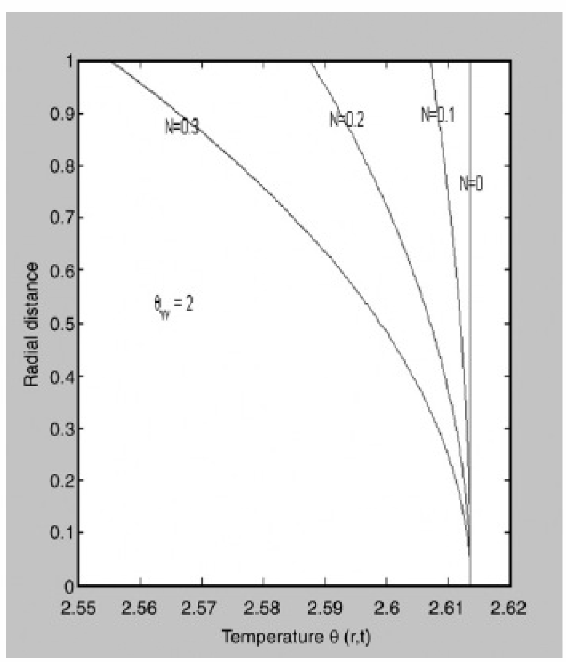

Fig. 3. Temperature distribution.

The shear stress τ is defined as

$$\tau = \mu\left(\frac{\partial u_0}{\partial r}\right)_{r=R_0} \tag{17}$$

On simplification we get,

$$\tau = \frac{1}{\mu^{\xi-2}}\left(\frac{R_0}{2}\right)^{\xi-1}\left[\frac{G_r^*\theta_\omega}{J_0\left(N\cdot\frac{R(x)}{R_0}\right)} - p_0\right]^{\xi-1}$$

4. Results and discussion

In this study we have analyzed pulsatile flow of a power law fluid as a model for blood flow in the cardiovascular system. We have invoked the differential approximation for the heat flux in the energy equation and taken the viscosity of blood as a function of the shear stress. In general the velocity is the Poiseuille velocity profile, but we can conclude from Fig. 2 that increase in the radiation parameter leads to an increase in the axial flow velocity, and increasing the exponential index, γ results in a reduction in the profile corresponding to a flattening of the peak as observed in Dash et al. [4]. When compared to with the work reported in Majhi and Nair [11], (A_g=0, Fig. 2a, that is no body acceleration), we see very good agreement between the two studies. The volume flow rate, Q is related to the velocity as shown in Eq. (16) and is not discussed here for brevity.

Fig. 3 shows the temperature distribution where we observe that the temperature decreases as the radiation parameter increases. We observe also that for any value of the radiation parameter the temperature at the vessel wall is lower than that along the axis, consistent with what obtains during deep heat muscle treatment (physiotherapy), Ogulu and Bestman [18].

Fig. 4 depicts how the shear stress τ is affected by the radiation parameter N, and the exponential index γ. In both cases we observe that the shear stress at the vessel wall is higher than that along the axis of the vessel. As the exponential index γ increases the shear stress is observed to decrease while an increase in the radiation parameter leads to an increase in shear stress, in qualitative agreement with the results in Ogulu [20].

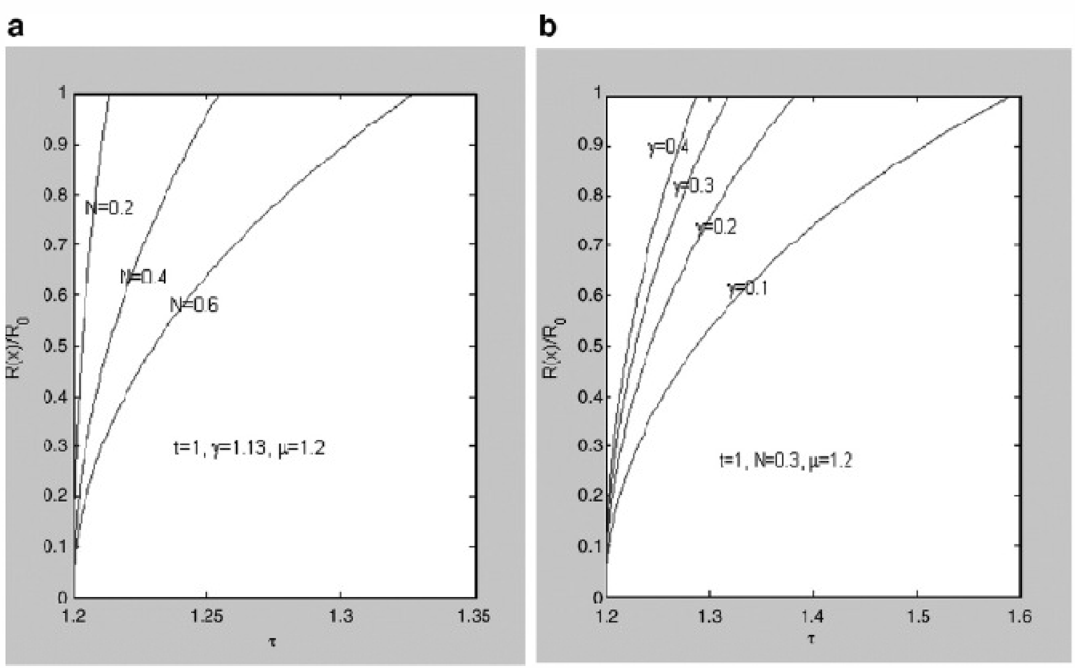

Fig. 4. (a). Effect of radiation on the shear stress distribution. (b). Effect of the exponential index on the shear stress distribution.

References

[1] S.A. Berger, L.D. Jou, Flow in stenotic vessels, Annual Review of Fluid Mechanics 32 (2000) 347–382.

[2] S. Chakravarty, A.K. Sannigrahi, An analytical estimate of the flow-field in a porous stenotic artery subject to body acceleration, International Journal of Engineering Science 36 (1998) 1083–1102.

[3] S.J. Yang, W.S. Fu, Numerical investigations of heat transfer from a heated oscillating rectangular cylinder in cross-flow, Numerical Heat Transfer. Part A, Applications 39 (6) (2001) 569–591.

[4] R.K. Dash, K.N. Mehta, G. Jayaraman, Casson fluid flow in a pipe filled with a homogenous porous medium, International Journal of Engineering Science 34 (10) (1996) 1145–1156.

[5] M. El-Shaded, Pulsatile flow of blood through a stenosed porous medium under periodic body acceleration, Applied Mathematics and Computation 138 (2003) 479–488.

[6] Y.L. Ju, L. Wang, Y. Zhou, Dynamic simulation of the oscillating flow with porous media in a pulse tube cryocooler, Numerical Heat Transfer. Part A, Applications 33 (7) (1998) 763–777.

[7] J.K. Misra, R.S. Chauhan, A study of pulsatile blood flow in a tube with pulsating walls, Acta Physica Hungarica 67 (1–2) (1990) 123–134.

[8] Z. Guo, H.J. Sung, J.M. Hyun, Pulsating flow and heat transfer in an annulus partially filled with porous media, Numerical Heat Transfer. Part A, Applications 31 (5) (1997) 517–527.

[9] A. Ogulu, T.M. Abbey, Simulation of heat transfer on an oscillatory blood flow in an indented porous artery, International Communications in Heat and Mass Transfer 32 (2005) 983–989.

[10] J.H. Moh, T.L. Bergman, D.C. Kuo, Simulation of two-dimensional, low-Pr natural convection in harmonically oscillated, differentially heat enclosures, Numerical Heat Transfer. Part A, Applications 31 (1) (1997) 1–17.

[11] S.N. Majhi, V.R. Nair, Pulsatile flow of third grade fluids under body acceleration-modelling blood flow, International Journal of Engineering Science 32 (5) (1994) 839–846.

[12] T. Tagawa, H. Ozoe, Effect of Prandtl number and computational schemes on the oscillatory natural convection in an enclosure, Numerical Heat Transfer. Part A, Applications 30 (3) (1996) 271–282.

[13] J. Prakash, O.D. Makinde, A. Ogulu, Magnetic effect on oscillatory blood flow in a constricted tube, Botswana Journal of Technology 13 (1) (2004) 45–50.

[14] T. Fusegi, Mixed convection in periodic open cavities with oscillating through flow, Numerical Heat Transfer. Part A, Applications 29 (1) (1996) 33–47.

[15] J.M. Fitz-Gerald, Non-Newtonian secretion flows in tubes, Journal of Fluid Mechanics 66 (3) (1974) 529–539.

[16] A. Ogulu, D.D.S. Bawo, Hydro-magnetic flow in a non-Newtonian material induced by an ultrasound beam, Modelling, Measurement and Control. B, Solid & Fluid Mechanics & Thermics, Mechanical Systems 71 (5–6) (1997) 17–26.

[17] S. Sekar, G. Nath, Study of molecular diffusion in oscillating flow in a pipe using boundary element method, Numerical Heat Transfer. Part A, Applications 20 (3) (1991) 345–366.

[18] A. Ogulu, A.R. Bestman, Deep heat muscle treatment: a mathematical model I, Acta Physica Hungarica 73 (1) (1993) 3–16.

[19] A.C.L. Cogley, W.G. Vincenti, E.S. Gilles, Differential approximation for radiative heat transfer in a nonlinear equations-grey gas near equilibrium, AIAAJ 6 (1968) 551–553.

[20] A. Ogulu, Effect of heat generation on low Reynolds number fluid and mass transport in a single lymphatic blood vessel with uniform magnetic field, International Communications in Heat and Mass Transfer 33 (2006) 790–799.