OPTIMAL INVESTMENT UNDER INFLATION PROTECTION AND

OPTIMAL PORTFOLIO WITH STOCHASTIC WAGE INCOME

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Optimal Investment Under Inflation Protection and Optimal Portfolio with Stochastic Wage Income

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Approval

This Dissertation has been examined and approved as meeting the requirements for the fulfilment of Master of Science Mathematics.

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Declaration

I, Hassan Kablay, hereby declare that the work contained in this dissertation is a guided work, and that any work done by others has been acknowledged and referenced accordingly. It is submitted in the partial fulfilment of the requirements for the award of the degree of Masters of Science (Mathematics) at the University of Botswana.

Signature .... Date................................

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Dedication

I dedicate this project to my late grandparents.

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Abstract

This dissertation considers the optimal portfolio strategies for an investor that obtains stochastic income and gives out stochastic cash outflows under inflation protection. The Investor trades on a complete diffusion model, receives a stochastic wage income and pays a stochastic cash outflow to its holder. The stochastic income is invested into a market that is characterized by a cash account, an inflation linked bond and a stock. The inflation risks associated with the investment could be hedged by investing in inflation-linked bond. The solutions to the Investor problem of seeking the optimal portfolio are formulated and worked out as stochastic control problem. The cash account is deterministic, and the inflation-linked bond and the stock are geometric. The optimal portfolio strategies for this Investor are solved and the utility function considered is assumed to be a quassi-concave function of the value of wealth and power utility is utilized. The optimal portfolio of the Investor in the cash account, in the inflation linked bond, and in the stock market were established.

Contents

Chapter 1

Introduction

This subject called mathematics is used in many fields like finance, economics, science and engineering. Mathematics is very broad and ranges from fields of pure mathematics to fields of applied mathematics. In applied mathematics, which we are interested in, for example, we have branches such as, mathematics of finance. The branch mostly utilizes the knowledge of Stochastic Differential Equations (SDEs), for the calculation of optimal portfolios. Mathematics of finance, a very important tool in finance, is also used in the modelling of option prices.

Taking the work of (Nkeki [17]) as a starting point, which is concerned with finding optimal portfolio and investment strategies for an investment company who received continuous time stochastic cash inflows and pays continuously a stochastic cash outflows to its holder, in our research we are focused on finding the optimal portfolios

and investment strategies for an Investor who received continuous stochastic wage income and pays continuously a stochastic cash outflow to its holder. The stochastic wage income is invested into cash account, an inflation-linked bond and a stock. Inflation-linked bonds are fixed income securities whose principal value is periodically adjusted according to the rate of inflation. According to (Nkeki [17]), inflation linked bonds are generally less risky than stocks, as they attract interest at a predetermined rate and have guaranteed returns. These inflation-linked bonds can be used to hedge inflation risk. (Nkeki [17]) went on and said, inflation risk is of increase and due to this increase in inflation risk in nations economy, investment companies have started investing optimally, the inflows paid by the holders into inflation linked bonds. In our research, the stochastic income will be invested in inflation linked bonds.

In related literature, (Nkeki [16]) studied an optimal portfolio strategy problem with discounted stochastic cash inflows. The analytical solution of the resulting HJB equation was found. Also, they found out that the smaller the level of risk the investor is willing to take, the higher the expected value of wealth, and vice versa. In addition to these findings, the optimal portfolio values in stock, inflation linked bond and cash account were obtained. Lastly, the resulting optimal portfolio values in stock and inflation-linked bond were found to involve intertemporal hedging terms that offset any shock to the stochastic cash inflows.

(Wu [26]) derived the analytical expression of the optimal investment strategy under inflation.Further, (Wu [26]) found out how risk aversion, correlation coefficient between inflation and the stock price, the inflation parameters and the coefficient of utility affect the optimal investment and consumption strategy. In our work, we consider the risk aversion tool as our power utility function.

Nkeki [15]) considered the mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. The optimal wealth involving a cash account and two risky assets of the pension plan member (PPM) were established. Also, efficient frontier for three asset classes which gives the PPM the opportunity to decide his or her own risk and wealth were obtained. Finally, it was found out that the inflation-linked bond is suitable for hedging inflation risks in an investment portfolio. (Battocchio [1]) studied a stochastic model for a defined-contribution pension fund in continuous time. The closed form solution for the set allocation problem was found and the behaviour of the optimal portfolio with respect to salary and inflation were analyzed in detail. In this research, we assume that the underlying assets, stochastic wage income and cash outflows are driven by a geometric Brownian motion.

1.0.1   Overview of Dissertation

The next sections of this dissertation will be discussed as follows: Chapter 2 will be the literature review on portfolio theory and preliminary concepts, and Chapter 3 will be the preliminary concepts. This will be followed by, Chapter 4 which will be stochastic control theory, then Chapter 5 will be the model and optimal portfolio strategies. Chapter 6 will be discussion and analysis of the results.

1.0.2   Objectives

We would like:

1. To calculate the discounted stochastic wage income process and discounted cash outflows process.

2. To find the value of the wealth process of the Investor at time t.

3. To find the optimal portfolio strategies for the Investor.

Chapter 2

Literature Review

2.1 Portfolio Theory

One of the key issues facing an individual is how to allocate wealth among alternative assets. Almost all financial institutions have the same problem with the added complication that they need to explicitly include the characteristics of their liabilities in the analysis. While the structure of these problems varies some what, they are similar enough that we classify both as portfolio theory (Elton et al [5]).

Mathematics, more specifically the Stochastic Differential Equations are greatly utilised in the calculation of how to allocate these assets.

Consider a consumer with a given amount of income. Such a consumer typically faces two important economic decisions. First, how to allocate his or her current consump-

tion among goods and services. Second, how to invest among various assets. These two interrelated consumer household problems are known as the consumer-saving decision and the portfolio selection decision (Constantinides et al [3]).

In financial markets, there are basic risks for investors, one of the most basic risks is, the erosion of real return of the portfolio by inflation (Yu et al[11]).

The whole idea underlying portfolio optimisation is totally natural. One has got a certain amount of money and tries to use it in such a way that one can draw the maximum possible utility from the results of the corresponding activities. This principle covers nearly every situation of daily life. Imagine that you are thinking of buying a house and are offered two different ones you can afford. One close to your office with public transport connections, but without a garden and close to a crowded motor way, the other one with a beautiful landscape but requiring you to commute a long distance to work everyday. The decision is about which is more convenient for you is in principle a portfolio problem (Korn [9])

A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies. The investor should build toward an integrated portfolio which best suits his needs.

A portfolio Analysis starts with information concerning individual securities. It ends with conclusions concerning portfolios as a whole. The purpose of the analysis is to find portfolios which best meet the objectives of the investor.

Various types of information concerning securities can be used as the raw material of a portfolio analysis. One source of information is the past performance of individual securities. A second source of information is the belief of one or more security analysts concerning future performances.

When past performances of securities are use as inputs, the outputs of the analysts are portfolios which performed well in the past. When beliefs of security analysis are used as inputs, the outputs of the analysis are the implications of these beliefs for better and worse portfolios (Markowitz [12]).

Some investors invest money in the financial market. Examples of such investors are banks, investment funds and insurance companies. Because these are serious investors, they always consider the risk involved even though they want to make as much money as possible. Normally, an investor is to a certain degree risk averse. For example, because of their obligations towards their customers, a traditional bank or an insurance company which invest funds on behalf of their customers in the financial market cannot allow themselves to take too much risk. The aim of such investors is to maximise the expected returns on their investments while at the same time limiting the risk involved. The theory of stochastic control and the maximisation of expected utility can be used to model such behaviour.

2.2   Utility Functions

A Utility Function is a mapping from the set of consumption choices into the real numbers. Simply, utility functions are a way of measuring an investors preferences for wealth and the amount of risks they are willing to undertake in the hope of attaining greater wealth. Utility functions can either be in Multiperiod Discrete-Time models or in Continous-Time Models.

The utility function expresses the preferences of economic entities with respect to perceived risk and expected return. Outcomes are transacted into numbers by the use of utility functions such that the expected value of the utility numbers can be used to calculate certainty equivalents for alternatives in such a way that is consistent with decision makers attitude towards risk-taking. The certainty equivalent for any gamble or alternative is the certain amount of money which gives the consumer exactly the same utility as the gamble/alternative. The risk premium of any gamble or alternative is the difference between the exected value of gamble/alternative and its certain equivalent, that is

$$\text{Risk Premium} = E(g) - C$$

where

$$g = \text{gamble}$$

and

$$C = \text{Certainty Equivalent.}$$

The certain equivalent C, for an investment whose outcome is given by a random variable X is:

$$C = U^{-1} \times E[U(X)]$$

which implies that

$$U(C) = E[U(X)]$$

(Cvitanić [4])

According to (Cvitanić [4]), utility functions $U(x)$ are such that

(i) U is strictly increasing. Therefore, maximising utility is equivalent to maximising the certainty equivalent.

(ii) U is twice differentiable.

(ii) U is concave. In particular,

$$U'' \leq 0$$

The certainty equivalent is always less than expected value of investment. In addition, utility functions are twice differentiable functions of wealth $U(w)$, $[w > 0]$, such that U is non-satiation (first derivative $[U' > 0]$) and risk aversion (second derivative $[U'' \leq 0]$). The value of an investment can be measured by the expected value of the utility of its consequence, and the largest expected utility is most preferable (Cvitanić

[4]).

In many investments, the consequences correspond to the investor receiving a certain amount of money. Let $U(x)$ be the investors utility of receiving amount x. $U(X)$ is therefore a utility function. If an investor must choose between two investments, of which the first returns an amount x and the second an amount y,then the investor should choose the first if

$$[\mathbb{E}[U(x)] > \mathbb{E}[U(y)]]$$

and the second if the inequality is reversed, where U is the utility function of that investor.

Definition 2.2.1. Marginal Utility is the additional satisfaction a consumer gains from consuming one or more unit of a good or service. This concept is very important because it is used to determine how much of an item a consumer will buy. There are several types of marginal utility. Below are some of them:

- Positive marginal utility is when the consumption of an additional item increases the total utility.

- Negative marginal utility is when the consumption of an additional item decreases the total utility.

- Zero marginal utility is when the consumption of an additional item does not

change the total utility.

2.2.1   Risk Preferences

Different Investors have different preferences for risk.

- Risk Averse has diminishing marginal utility of wealth.

- Risk Neutral has constant marginal utility of wealth.

- Risk Lover has increasing marginal utility of wealth.

The value of an investment can be measured by the expected value of the utility function of its consequence, and the investment with the largest expected utility is most preferable.

Let U(x) be the utility function(the investors utility of receiving the amount x). Therefore,if an investor must choose between two investments, of which the first returns an amount X and the second an amount Y, then the investor should choose the first if

$$\mathbb{E}[U(X)] > \mathbb{E}[U(Y)]$$

and the second if the inequality is reversed.

An investor's utility function is specific to that investor. Also, a general property usually assumed of utility functions is that $U(x)$ is a non-decreasing function of x. A common feature for most investors is that, if they expect to receive x, then the extra utility gained if they are given an additional amount Δ is non-increasing in x, that

is, for fixed $\Delta > 0$, their utility function satisfies

$$U(x + \Delta) - U(x)$$

is non-increasing in X. A utility function that satisfies this condition is called con-cave. Also, it can be shown that the condition of concavity is equivalent to $U"(x) \leq 0$. This simply means that, a function is concave if and only if its second derivative is non-positive(Ross [21]).

2.2.2 Risk Averse Investor

Definition 2.2.2. A Risk-averse Investor is an investor with a concave utility func-tion. Jensen's inequality is the one that gives such a investor the name risk-averse investor. Jensen's inequality states that if u is a concave function, then for any random variable X,

$$\mathbb{E}[U(X)] \leq U(\mathbb{E}[X])$$

. By Jensen's inequality and letting X be the return from an investment, any investor with a concave utility function would prefer the certain return of $\mathbb{E}[X]$ to receiving a random return with this mean (Ross [21]).

The above inequality simply means that; the investor prefers the certain average amount $\mathbb{E}(X)$ to the random amount X. The utility function for a risk averse in-dividual must be concave (from below) such that the chord lies below the utility

function (Cvitanić [4]).

Another utility function that is commonly used is the log utility function

$$U(X) = \log(X).$$

$\log(X)$ is a concave function, therefore, an investor with a log utility function is risk-averse. In a variety of situations an investor with an infinite sequence of investments can maximize long-term rate of return by adopting a log utility function and then maximizing the expected utility in each period. This can be proven mathematically. This makes the log utility function very important (Ross [21]).

2.2.3 Other Examples of Utility Functions

- Logarithmic Utility:

$$U(x) = \log(x)$$

- Power Utility :

$$U(x) = \frac{x^{\gamma}}{\gamma}, \ \gamma \leq 1$$

- Exponential Utility:

$$U(x) = 1 - \exp(-\alpha x)$$

- Quadratic Utility:

$$U(x) = x - \beta x^2$$

(Cvitanić [4])

In our dissertation, we choose the power utility function with the parameter γ as the

risk-aversion parameter.

Chapter 3

Preliminary Concepts

3.1 Introduction

In this chapter, we define the preliminary concepts thats will be used. This includes their definitions and some examples.

3.1.1 Sample Space

Definition 3.1.1. A sample space is the set of all possible outcomes of an experiment.

Example 3.1.1. If a die is thrown once, the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$. This means that each time the die is thrown, a 1,2,3,4,5 or 6 can be obtained.

The results of a random experiment are modelled by this set, where each point of Ω

corresponds to all possible outcome.

Example 3.1.2. If we throw pair of dice, the sample space could consist of the 36 ordered pairs (a,b) where a and b take values from 1 to 6 (Promislow [20]).

3.1.2 σ - algebra

Definition 3.1.2. Let Ω be a given set. Then a σ - algebra, $\mathcal{F}$, (also known as a σ-field) on a Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

   (i) $\emptyset \in \mathcal{F}$.

   (ii) $F \in \mathcal{F} \Longrightarrow F^c \in \mathcal{F}$, where $F^c = \Omega \backslash F$ is the complement of F in Ω.

   (iii) $A_1, A_2, ... \in \mathcal{F} \Longrightarrow A := \bigcup_{i=0}^{\infty} A_i \in \mathcal{F}$.

Example 3.1.3. Let $\Omega = \{1, 2, 3\}$ and $\mathcal{F} = \{\{1, 2\}, \{3, \}, \{1, 2, 3\}\}$ be a family of subsets. Then:

   (i) $\emptyset \in \mathcal{F}$ as $\emptyset$ is in any set.

   (ii) Now to prove the second property of σ - algebra, we pick each element in $\mathcal{F}$
        and see if its complement is also in $\mathcal{F}$; $\{1, 2\} \in \mathcal{F}$, then $\{1, 2\}^c = \{3\} \in \mathcal{F}$,
        $\{3\} \in \mathcal{F}$ then $\{3\}^c = \{1, 2\} \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$ then $\emptyset^c = \{1, 2, 3\} \in \mathcal{F}$, and also,
        $\{1, 2, 3\} \in \mathcal{F}$ then $\{1, 2, 3\}^c = \emptyset \in \mathcal{F}$.

(iii) Finally, to prove the third property of σ - algebra: If $\{1,2\},\{3\},\{1,2,3\} \in$

$$\mathcal{F} \implies \{1,2\}\bigcup\{3\}\bigcup\{1,2,3\} = \{1,2,3\} \in \mathcal{F}.$$

The above shows that all the conditions of σ - algebra are satisfied, hence $\mathcal{F}$ is a σ - algebra.

In a market situation, the σ- algebra $\mathcal{F}_t$ models the information that is revealed to an investor at time t.

3.1.3 Measurable Space

Definition 3.1.3. A measurable space is a pair $(\Omega, \mathcal{F})$ where Ω is a sample space and $\mathcal{F}$ is a σ - algebra defined on $(\Omega, \mathcal{F})$.

3.1.4 Event

An event is a combination of outcomes. This is represented by a subset of Ω. Such an event occurs if any of the outcomes in the subset occur (Promislow[20]).

Example 3.1.4. In an experiment of tossing a pair of die, $\Omega = \{1,2,3,4,5,6\}$, $A = \{3,2\}$. A is an event, meaning that it is possible to obtain a 3 and a 2 at the same time by tossing the pair of die.

3.1.5   Probability

Definition 3.1.4. Probability measure: A probability measure $\mathbb{P}$ on a measurable

space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ such that:

(a) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

(b) if $A_1, A_2 ...... \in \mathcal{F}$ and A_i is disjoint, that is, $A_i \bigcap\limits_{i=1}^{\infty} A_j = \emptyset$ if $i \neq j$,   then

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

From the definition, a probability is a function or a map with the domain σ-algebra,

the codomain $\mathbb{R}$ and the range $[0,1]$.

3.1.6   Probability Space

Definition 3.1.5. A probability space is a 3-tuple, $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a sample

space, $\mathcal{F}$ is a σ-algebra defined on Ω and $\mathbb{P}$ is a probability measure defined on $\mathcal{F}$.

3.1.7   Filtration

Definition 3.1.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is

an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub-σ - algebras of $\mathcal{F}$.

In other words, for each t, $(\mathcal{F}_t)_t \geq 0$ is a σ - algebra included in $(\mathcal{F}_t)$ and if $s \leq t$,

$(\mathcal{F}_s) \subset (\mathcal{F}_t)$. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$ is called a filtered probability space.

Further, A filtration represents an increasing stream of information. In a market, share prices, exchange rates, interest rates e.t.c can be modelled by solutions of stochastic differential equations which are driven by Brownian motion. These solutions are then functions of Brownian motion. The fluctuations of these processes actually represent the information about the market. This relevant knowledge is contained in the natural filtration (Mikosch [13]).

In our case, the filtration model will be used to model the information that is revealed to an investor. In a market situation, the increasing nature of σ - algebra shows that information is never forgotten. The information we have time t is the sum of the accumulation of information from t.

3.1.8 Random Variable

Definition 3.1.7. A Random variable is a function that associates a real number with each element in the sample space. That is, $X : \Omega \to \mathbb{R}$. We usually use capital letters to denote a random variable and the corresponding small letter to denote a value of the random variable.

Example 3.1.5. Tossing two dice.

$$\Omega = \{(i,j) : i, j = 1, 2, ...6\}$$

. There are several random variables that can be defined, for example,

$$X = i + j, \quad Y = |i - j|$$

Both X and Y are random variables. X can take values $2, 3, ..., 12$ and Y can take values $0, 1, ..., 5$.

Basically, a random variable is a function that maps outcomes to real numbers.

Types of Random variables

There are two types of random variables. The discrete random variable and the continuous random variable.

The discrete random variable represents distinct values, for example, $X = 0, 1, 2, ...,$ while the continuous random variable represents continuous values, for example, $X > 3$, which could be used in Height of plant, Time e.t.c.

3.1.9   Expectation

Definition 3.1.8. The expectation (also known as the mean) of a random variable X, represents in some sense the average value that X will take.

For a discrete random variable, let $X = \{1, 2, ..., k\}$, where $X_k = k$ and $\mathbb{P}(X = k) = f(k)$. Then:

$$\mathbb{E}(X) = \Sigma_{k=1}^{\infty} k f(k),$$

While for a continuous random variable

$$\mathbb{E}(X) = \int_0^{\infty} x f(x) dx.$$

If g is a function defined on a set that includes the range of X, we can define another random variable $g(X)$ that takes value $g(x)$ when X takes the value x. The expectation of this random variable is given by:

$$\mathbb{E}[g(X)] = \sum_{k=0}^{\infty} g(k) f(k)$$

OR

$$\mathbb{E}[g(X)] = \int_0^{\infty} g(x) f(x) dx$$

depending on whether X is discrete or continuous.

Of particular importance are the functions $g(x) = x^n$. For such a function $\mathbb{E}[g(x)]$ is known as the nth moment of X.

Definition 3.1.9. The variance of X is defined by:

$$
\begin{aligned}
Var(X) &= \mathbb{E}[X - \mathbb{E}(X)]^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X)^2
\end{aligned}
$$

The square-root of $Var(X)$ is known as the standard deviation of X.

The smaller the variance, the more likely it is that values of X are close to the mean.

A formal statement along these lines is given by Chebyshev's inequality, which states

that for $k > 0$,

$$
\mathbb{P}(\parallel X - \mathbb{E} \parallel \geq k) \leq \frac{Var(X)}{k^2}
$$

(Promslow[20])

According to (Øksendal [18]), let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a given complete probability space.

Let X be a random variable that is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$.

If $\int_{\Omega} |X(\omega)| dP(\omega) < \infty$, then the number:

$$
\mathbb{E}[X] : = \int_{\Omega} X(\omega) dP(\omega)
$$

$$= \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called the expectation of X (*w.r.t* $\mathbb{P}$).

3.1.10 Stochastic Process

Definition 3.1.10. A stochastic process is a family (X_t) of real valued random variables indexed by time. It is continuous if $t \to X_t(w)$ is continuous almost surely. A stochastic process (X_t) is adapted to the filtration $(\mathcal{F}_t)$ if for every $s \in [0, T]$, the random variable X_s is $\mathcal{F}_t$ measurable.

3.1.11 Standard Brownian Motion

Brownian motion is one of the stochastic process which is used to model noise in a market.

The dominion of financial asset pricing borrows a great deal from the field of stochastic calculus. The price of a stock tends to follow a Brownian motion ([22]). The mathematical foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, and this process is also called the Wiener Process. The Brownian motion process $W(t)$ serves as a Basic model for the cumulative effect of pure noise(klebaner [8]).

Definition 3.1.11. A standard Brownian motion (or standard Wiener process) is a stochastic process $W = \{W_t\}_{t \geq 0}$, i.e. a collection of random variables W_t defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the following conditions:

1. $W_0 = 0$,

2. with probability one, the function W_t is continuous in t,

3. W has stationery and independent increments, i.e. for any positive integer n and any $0 = t_0 < t_1 < .... < t_n$, the random variables $W_t - W_{ti-1}$, $i = 1, ..., n$ are mutually independent , and $W_{s+t} - W_s$ has the same distribution as W_t for any s, $t > 0$.

4. $W_t \sim N(0, T)$

In this dissertation, Brownian motion will be used as a model for stock price behaviour.

Proposition 3.1.1. If Y and Z are stochastic variables, and Z is $\mathcal{F}_t$-measurable, then:

$$\mathbb{E}[Z.Y|\mathcal{F}_t] = Z.\mathbb{E}[Y|\mathcal{F}_t]$$

In the expected value $\mathbb{E}[Z.Y|\mathcal{F}_t]$, we condition upon all information available at time t. If now $Z \in \mathcal{F}_t$, this means that, given the information $\mathcal{F}_t$, we know exactly the value of Z, so in the conditional expectation Z can be treated as a constant, and thus it can be taken outside the expectation (Bjork [2]).

Proposition 3.1.2. If Y is a stochastic variable, and if $s < t$, then

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s]$$

This result is called the "law of iterated expectation", and it is basically a version of the law of total probability (Bjork [2]).

3.1.12   Geometric Brownian Motion

Definition 3.1.12. A non-negative variation of Brownian Motion called geometric Brownian Motion $S(t)$ is defined by

$$S(t) = s_0 \exp\big(X(t)\big)$$

where

$$X(t) = \sigma W(t) + \mu t$$

is Brownian Motion with drift and

$$S(0) = s_0 > 0$$

is the initial value. Geometric brownian motion is used for modelling stock prices.

3.1.13   Market

Definition 3.1.13. A market is an $\mathcal{F}_t^{(m)}$ -adapted $(n+1)$ -dimensional Itô process

$$X(t) = (X_0(t), X_1(t), ..., X_n(t)); \quad 0 \leq t \leq T$$

whose dynamics is represented by a system of differential equations.

$$
\begin{aligned}
dX_0(t) &= \rho(t,\omega)X_0(t)dt; \quad X_0(0) = 1 \quad\quad (3.1)\\
dX_i(t) &= \mu_i(t,\omega)dt + \Sigma\sigma_{i,j}(t,\omega)dW_j(t)\\
&= \mu_i(t,\omega)dt + \sigma_i(t,\omega)dW(t); \quad X_i(0) = x_i
\end{aligned}
$$

where σ_i is row number i of the $n \times m$ matrix $[\sigma_{i,j}]$; $\quad 1 \leq i \leq n \in \mathbb{N}$.

Furthermore, the market $X(t) \in [0, T]$ is called normalized if $X_0(t) = 1$

(Øksendal [18]).

Basically, $X_i(t) = X_i(t, \omega)$ is taken as the price of security/asset number i at time t. The assets number $1, ..., n$ are called risky because of their diffusion terms. For example, they can represent stock investment.

Asset number 0 is called safe because of the absence of the diffusion term, for example, Bank account. Assume $\rho(t, \omega)$ is bounded for simplicity.

Normalised Market

Definition 3.1.14. $\overline{X}(t) = (1, \overline{X}_1(t), ..., \overline{X}_n(t))$ is the normalization of $X(t)$ and is simply derived by dividing each of the assets by the price of the safe investment

Portfolio

Definition 3.1.15. A portfolio in the market $X(t)_{t\in[0,T]}$ is an $(n+1)$ dimensional (t,ω) -measurable and $\mathcal{F}_t^{(m)}$ -adapted stochastic process $\theta(t,\omega) = (\theta_0(t,\omega), \theta_1(t,\omega), ..., \theta_n(t,\omega))$; $0 \leq t \leq T$. A time t of a portfolio $\theta(t)$, the value at this time is defined as

$$V(t,\omega) = V^\theta(t,\omega) = \theta(t) \cdot X(t) = \sum \theta_i(t)X_i(t)$$

where $\cdot$ denotes the inner product $\mathbb{R}^{n+1}$.

Basically, a portfolio is a particular combination of assets in question. To form a portfolio one needs to know the positions taken in each asset under consideration. The symbol θ_i represents the commitment with respect to the i_{th} assets. Specifying all $\theta_i; i = 1, ..., N$ specifies the portfolio.

A positive θ_i implies a long position in that asset, while a negative θ_i implies a short position. If an asset is not included in the portfolio, the corresponding θ_i is zero. If a portfolio delivers the same pay-off in all states of the world, then its value is known exactly and the portfolio is risk less (Neftci [14]).

Self-financing Portfolio

Definition 3.1.16. The portfolio $\theta(t)$ is called self-financing if

$$\int_0^T \left[|\theta_0(s)\rho(s)X_0(s) + \Sigma_{i=1}^n \theta_i(s)\mu_i(s)| + \Sigma_{j=1}^m [\Sigma_{i=1}^n \theta_i(s)\sigma_{i,j}(s)]^2 \right] ds < \infty \quad \text{a.s.}$$

and

$$dV(t) = \theta(t).dX(t)$$

.

That is,

$$V(t) = V(0) + \int_0^t \theta(s).dX(s) \quad \text{for t} \quad \in [0,T]$$

(Øksendal [18])

Definition 3.1.17. A portfolio $\theta(t)$ which is self-financing is called admissible if the corresponding value process $V^\theta(t)$ is (t,ω) a.s. lower bounded, that is,, there exists $K = K(\theta) < \infty$ such that:

$$V^\theta(t,\omega) \geq -K \quad \text{for a.a.} \quad (t,\omega) \in [0,T] \times \Omega \tag{3.2}$$

This restriction (3.2) reflects a natural condition in real life finance. There must be a limit to how much debt the creditors can tolerate. ([18])

Definition 3.1.18. An admissible portfolio $\theta(t)$ is called arbitrage in the market $X_{tt\in[0,T]}$ if the corresponding value process $V^\theta(t)$ satisfies

$$V^\theta(0) = 0 \quad \text{and} \quad V^\theta(T) \geq 0 \quad \text{a.s.} \tag{3.3}$$

$$\text{and} \quad \mathbb{P}[V^\theta(T) > 0] > 0. \tag{3.4}$$

(Øksendal [18])

3.1.14 Martingales

Martingales are one of the central tools in the modern theory of finance. Martingale theory classifies observed time series according to the way they "trend". A stochastic process behaves like a martingale if its trajectories display no discernible trends or periodicities (Neftci [14]).

Continous-time Martingales

Definition 3.1.19. Let $\mathcal{M}_t$ be a filtration.

An n-dimensional stochastic process $(M_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $(\mathcal{M}_t)_{t\geq 0}$ (and with respect to $\mathbb{P}$) if:

(i) M_t is $\mathcal{M}_t$-measurable for all t.

(ii) $\mathbb{E}[|\ M_t\ |] < \infty$ for all t.

(iii) $\mathbb{E}[M_s|\mathcal{M}_t] = M_t$ for all $s \geq t$.

The expectation in (ii) and the conditional expectation in (iii) are taken with respect to $\mathbb{P} = \mathbb{P}^0$ (Øksendal[18])

According to (Neftci [14]), property (iii) above implies that the best forecast of un-observed future values is the last observation on M_t.

Martingales are stochastic processes whose dynamics are completely unpredictable. The best forecast for the value of the process at time s given information at time t is the value of the process at time t.

(Neftci [14]) further went on to say that, martingales are random variables whose future variations are completely unpredictable given the current information set. The best forecast of the change in M_t over an arbitrary interval $t > 0$ is zero. In other words, the directions of the future movements in martingales are impossible to forecast. This is the fundamental characteristics of processes that behave as martin-gales. If the trajectories of a process display clearly recognizable long-or short run

"*trends*", then the process is not a martingale.

It is also important to take note that a martingale is always defined with respect to some information set, and with respect to some probability measure. If we change the information content and /or the probabilities associated with the process, the process under consideration may cease to be a martingale. The opposite is also true. Given a process X_t which does not behave like a martingale , we may be able to modify the relevant probability measure $\mathbb{P}$ and convert X_t into a martingale (Neftci [14]).

This property is very important in finance and in our project. Stock price processes are not martingale since on average they are increasing in nature. But when pricing using a martingale approach, it is required that the process is a martingale. This is achieved by changing the probability measure. In other words, we find an equivalent probability such that the process becomes a martingale.

According to the definition of the martingale, a process X_t is a martingale if its future movements are completely unpredictable given a family of information sets. Now, we know that stock prices or bond prices are not completely unpredictable. The price of a discount bond is expected to increase over time. In general, the same is true for stock prices. They are expected to increase on the average (Neftci [14]).

Equivalent Martingale Measure

Definition 3.1.20. Let $(\Omega, \mathcal{F}, (\mathcal{F})_{n \geq 1}, \mathbb{P})$ be a filtered probability space. Then we say that a measure $\widetilde{\mathbb{P}}$ in $(\Omega, \mathcal{F})$, is absolutely continous with respect to $\mathbb{P}$ (we write $\widetilde{\mathbb{P}} \ll \mathbb{P}$ if $\widetilde{\mathbb{P}}(A) = 0$ for each $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$).

Definition 3.1.21. The measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ in the same measurable space $(\Omega, \mathcal{F})$ are equivalent (we write $\widetilde{\mathbb{P}} \sim \mathbb{P}$) if $\widetilde{\mathbb{P}} \ll \mathbb{P}$ and $\mathbb{P} \ll \widetilde{\mathbb{P}}$.

Let $\widetilde{\mathbb{P}} \ll \mathbb{P}$. Then $\widetilde{\mathbb{P}}_n \ll \mathbb{P}_n$ for each $n \in \mathbb{N}$ and there exist Radon-Nikodyn derivatives denoted by

$$\frac{d\widetilde{\mathbb{P}}_n}{d\mathbb{P}_n} \quad or \quad \frac{d\widetilde{\mathbb{P}}_n}{d\mathbb{P}_n}(\omega)$$

and defined as $\mathcal{F}_n$-measurable functions $Z_n = Z_n(\omega)$ such that

$$\widetilde{\mathbb{P}}_n(A) = \int_A Z_n(\omega)\mathbb{P}_n(d\omega), \quad A \in \mathcal{F}_n.$$

(Shiryaev [23])

3.1.15   Itô Integral for elementary functions

Definition 3.1.22. Let $\mathcal{V} = \mathcal{V}(S, T)$ be the class of functions

$f(t, \omega) : [0, \infty) \times \Omega \implies \mathbb{R}$ such that:

(i) $(t, \omega) \longrightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ -measurable, where $\mathcal{B}$ denotes the Borel σ - algebra on $[0, \infty)$.

(ii) $f(t, \omega)$ is $\mathcal{F}_t$ adapted.

(iii) $\mathbb{E}[\int_S^T f(t, \omega)^2 dt] < \infty$

For functions $f \in \mathcal{V}$ we will now define the Itô integral

$\mathcal{I}[f](\omega) = \int_S^T f(t, \omega) dW_t(\omega)$, where W_t is 1-dimensional Brownian motion.

Definition 3.1.23. A function $\phi \in \mathcal{V}$ is defined as elementary if it has the form

$$\phi(t, \omega) = \Sigma_j e_j(\omega).\chi_{t_j, t_{j+1}}(t)$$

where χ denotes the characteristic (indicator) function where $\phi \in \mathcal{V}$ and ϕ is an elementary function, e_j are constants, $j \in \mathbb{N}$.

Since $\phi \in \mathcal{V}$ each function e_j must be $\mathcal{F}_{t_j}$ -measurable.

.

The Itô integral $\int_S^T f(t, \omega) dW_t(\omega)$ where (f $\in \mathcal{V}$, W_t is a 1-dimensional Brownian Motion), can be defined as

$$\int_S^T \phi(t, \omega) dW_t(\omega) = \Sigma_{j \geq 0} e_j(\omega)[W_{t_{j+1}} - W_{t_j}](\omega)$$

The Itô Isometry

Definition 3.1.24. If $\phi(t, \omega)$ is bounded and elementary, then:

$$\mathbb{E}\left[(\int_S^T \phi(t, \omega)dW_t(\omega))^2\right] = \mathbb{E}\left[\int_S^T \phi(t, \omega)^2 dt\right] \qquad (3.5)$$

Proof. Refer to (Øksendal [18]).

$\square$

Itô Integral for functions in $\mathcal{V}$.

The isometry (3.5) was used to extend the definition of It$\widehat{o}$ integral from elementary functions in $\mathcal{V}$.

Definition 3.1.25. (The Itô Integral)

Let $f \in \mathcal{V}(S, T)$. Then the It$\widehat{o}$ integral of f (from S to T) is defined by

$$\int_S^T f(t, \omega)dW_t(\omega) = \lim_{n \to \infty} \int_S^T \phi_n(t, \omega)dW_t(\omega) \quad \text{(limit in } L^2(P))$$

where ϕ_n is a sequence of elementary function such that

$$\mathbb{E}\left[\int_S^T (f(t, \omega) - \phi_n(t, \omega))^2 dt\right] \to 0 \quad \text{as} \ \ n \to \infty$$

Corollary 3.1.26. (The Itô Isometry)

$$\mathbb{E}\left[\left(\int_S^T f(t,\omega)dW_t\right)^2\right] = \mathbb{E}\left[\int_S^T f^2(t,\omega)dt\right] \quad \text{for all} \quad f \in \nu(S,T)$$

(Øksendal [18])

Theorem 3.1.27. Let $f, g \in \mathcal{V}(0,T)$ and let $0 \le S < U < T$. Then:

(i) $\int_S^T = \int_S^U f dW_t + \int_U^T \quad$ for almost all ω.

(ii) $\int_S^T (cf + g)dW_t = c \int_S^T f dW_t + \int_S^T g dW_t \quad$ for almost all ω, where c is a constant.

(iii) $\mathbb{E}[\int_S^T f dW_t] = 0$

(iv) $\int_S^T f dW_t$ is $\mathcal{F}_T$-measurable.

Proof. (Refer to Øksendal [18])

$\square$

An important property of the Itô integral is that it is a martingale.

3.1.16   Itô Formula

The basic definition of Itô integrals is not very useful when we try to evaluate a given integral. But, it is possible to establish an Itô integral version of the chain rule, known as, the Itô Formula.

We introduce Itô process (also called stochastic integrals) in order to make the family of integrals that are combination of the dB_s-integral and ds-integral stable under smooth maps.

Definition 3.1.28. (1-dimensional Itô Process).

Let W_t be the $1-$dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A ($1-$dimensional) Itô process(or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form:

$$X_t = X_0 + \int_0^t u(s, \omega)ds + \int_0^t v(s, \omega)dW_s \tag{3.6}$$

$\mathcal{W}_{\mathcal{H}}(S, T)$ denotes the class of process $f(t, \omega) \in \mathbb{R}$ satisfying:

(i) $f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$ such that $(t, \omega) \to f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable. where $\mathcal{B}$ denotes the Borel σ–algebra on $[0, \infty)$.

(ii) There exists an increasingly family of σ-algebras $\mathcal{H}_t$; $t \geq 0$ such that :

(a) B_t is a martingale with respect to $\mathcal{H}_t$,

(b) f_t is $\mathcal{H}_t$-adapted.

(iii) $\mathbb{P}[\int_S^T f(s,\omega)^2 ds < \infty] = 1$,

where $v \in \mathcal{W}_H$, so that

$$\mathbb{P}[\int_0^t v(s,\omega)^2 ds < \infty \ \text{ for all } \ t \geq 0] = 1$$

We also assume that U is $\mathcal{H}_t$-adapted and;

$$\mathbb{P}[\int_0^t \mid U(s,\omega) \mid ds < \infty \ \text{ for all } \ t \geq 0] = 1$$

If X_t is an Itô process of the form (3.6), then (3.6) is sometimes written in the shorter differential form

$$dX_t = udt + vdW_t$$

Theorem 3.1.29. (The $1-$dimensional Itô formula)

Let X_t be an Itô process given by:

$$dX_t = udt + vdW_t. \tag{3.7}$$

Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$, ($g$ is twice continuously differentiable on $[0, \infty) \times \mathbb{R}$).

Then,

$Y_t = g(t, X_t)$ is again an Itô process, and ;

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)(dX_t)^2$$

where

$$(dX_t)^2 = (dX_t)(dX_t) \tag{3.8}$$

is compared according to the rules

$$dt.dt = dt, dW_t = dW_t.dt = 0; dW_t.dW_t = dt$$

(Øksendal [18])

Example 3.1.6. Take the integral

$$I = \int_0^t W_s dW_s$$

Assume

$$W_0 = 0$$

Let $X_t = W_t$ and $g(t, x) = \frac{1}{2}x^2$.

Then, by Itôs formula,

$$
\begin{aligned}
dY_t &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dW_t \\
&\quad + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dW_t)^2 \\
&= 0.t + W_t dW_t + \frac{1}{2}(1)(dW_t)^2 \\
&= W_t dW_t + \frac{1}{2}dt
\end{aligned}
$$

Hence:

$$
\begin{aligned}
d(\frac{1}{2}W_t^2) &= W_t dW_t + \frac{1}{2}dt \\
\int_0^t d(\frac{1}{2}dW_s^2) &= \int_0^t W_s dW_s + \int_0^t \frac{1}{2}ds \\
\frac{1}{2}W_s^2 \mid_0^t &= \int_0^t W_s dW_s + \frac{1}{2}s \mid_0^t \\
\frac{1}{2}W_t^2 - \frac{1}{2}W_0^2 &= \int_0^t W_s dW_s + \frac{1}{2}s - \frac{1}{2}(0) \\
\frac{1}{2}W_t^2 &= \int_0^t W_s dW_s + \frac{1}{2}s
\end{aligned}
$$

(Øksendal [18])

Theorem 3.1.30. (The Itô Representation Theorem)

Let $F \in L^2(\mathcal{F}_T^{(n)}, \mathbb{P})$. Then, there exists n a unique stochastic process $f(t,\omega) \in \mathcal{V}(0,T)$ such that

$$F(\omega) = \mathbb{E}[F] + \int_0^T f(t,\omega)dW_{(}t)$$

Proof. Refer to (Øksendal [18]).

$\square$

3.1.17   The Brownian Martingale Representation Theorem

Definition 3.1.31. A $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale $\{M_t\}_{t\geq 0}$ is said to be square-integrable if

$$\mathbb{E}[|M_t|^2] < \infty$$

for each $t > 0$.

Theorem 3.1.32. Brownian Martingale Representation Theorem.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration of the $\mathbb{P}$-Brownian motion $\{B(t)\}_{t\geq 0}$. Let $\{M_t\}_{t\geq 0}$ be a square integrable $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale. Then there exists an $\{\mathcal{F}_t\}_{t\geq 0}$-predictable process $\{u_t\}_{t\geq 0}$ such that with $\mathbb{P}$-probability one,

$$M_t = M_0 + \int_0^t u_s dW(s).$$

(Etheridge [6])

Proof. Refer to (Etheridge [6]).

$\square$

3.1.18 The Girsanov Theorem

Theorem 3.1.33. Suppose that $\{W(t)\}_{t\geq 0}$ is a $\mathbb{P}$-Browian motion with natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and that $\{U(t)\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}U^2(t)dt\right)\right] < \infty.$$

Define

$$M_t = \exp\left(-\int_0^t u(s)dW(s) - \frac{1}{2}\int_0^t U^2(s)ds\right)$$

and let $\mathbb{P}^{(M)}$ be the probability measure defined by

$$\mathbb{P}^{(M)}[\Omega] = \int_\Omega M_t\mathbb{P}d(\omega)$$

. Then under the probability measure $\mathbb{P}^{(M)}$, the process $\{B^{(M)}(t)\}_{0\leq t\leq T}$, defined by

$$W^{(M)}(t) = W_t + \int_0^t U(s)ds$$

is a standard Brownian motion (Etheridge [6]).

3.1.19 Existence And Uniqueness Theorem For Stochastic Differential Equations

Let $T > 0$ and

$$b(.,.) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma(.,.) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$$

be measurable functions satisfying:

$$| b(t, x) | + | \sigma(t, x) | \leq C(1+ | x |); x \in \mathbb{R}^n, t \in [0, T] \tag{3.9}$$

for some constant C, (where $| \sigma |^2 = \sum | \sigma_{ij} |^2$) and such that;

$$| b(t, x) - b(t, y | + | \sigma(t, x) - \sigma(t, y) | < D | x - y |; x, y \in \mathbb{R}^n, t \in [0, T] \tag{3.10}$$

for some constant D.

Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $W_s(.)$, $s \geq 0$ and such that

$$\mathbb{E}[| Z |^2] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, x_t)dt + \sigma(t, x_t)dW_t, \quad 0 \leq t \leq T, \quad x_0 = Z$$

has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t$ generated by Z and

$$W_s(.); \quad s \leq t$$

and

$$\mathbb{E}[\int_0^T | X_t |^2 \, dt] < \infty.$$

Proof. Refer to (Øksendal [18]).

$\square$

3.1.20 Stochastic Differential Equations (SDEs)

Stochastic Differential Equations (SDEs)are widely used in Financial Modelling.

In applications, SDE is how we think about random process that evolve over time. For instance, the return on a portfolio. The idea is not that various physical phenomena are Brownian motion, but that they are driven by a Brownian motion.

Differential Equations are used to describe the evolution of a system. Stochastic Differential Equations arise when a random noise is introduced into Ordinary Differential Equations(ODEs).

Definition 3.1.34. If $x(t)$ is a differential function defined for $T \geq 0$, $\mu(x,t)$ is a function of x, and t, and the following relation is satisfied for all $t, 0 \leq t \leq T$;

$$\frac{dx(t)}{dt} = x'(t) = \mu(x(t), t) \quad \text{and} \quad x(0) = x_0, \qquad (3.11)$$

then $x(t)$ is a solution of the ODE with the linear condition x_0 (Klebaner [8]). Usually the requirement that $x'(t)$ is continuous is added.

White Noise and SDEs

The white noise process $\xi(t)$ is formally defined as the derivative of the Brownian motion;

$$\frac{dW(t)}{dt} = W'(t)$$

It does not exist as a function of t in the usual sense, since a Brownian motion is nowhere differentiable.

If $\sigma(t)(x, t)$ is the intensity of the noise at point x at time t, then it is agreed that;

$$\int_0^T \sigma(x(t), t)\xi(t)dt = \int_0^T \sigma(x(t), t)W'(t)dt = \int_0^T \sigma(x(t), t)dW(t), \qquad (3.12)$$

where the integral is Itô integral.

Stochastic Differential Equations arise, for example, when the coefficients of ordinary differential equations are perturbed by White Noise.

Definition 3.1.35. Let $W(t), t \geq 0$ be Brownian motion process. An equation of the form:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t),$$

where functions $\mu(x, t)$ and $\sigma(x, t)$ are given $X(t)$ is the unknown process, is called a Stochastic Differential Equation (SDE) driven by Brownian motion. The functions $\mu(x, t)$ and $\sigma(x, t)$ are called the coefficients(Klebaner [8]).

3.2 Stochastic Control

Let the state of the system at time t be described by an Itô process X_t of the form

$$dX_t = dX_t^u = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t$$

where

$$X_t \in \mathbb{R}^n, b : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : \mathbb{R} \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}$$

and W_t is m-dimensional Brownian motion. $u_t \in U \subset \mathbb{R}^k$ is a parameter whose value is in the Borel set U at any instant t in order to control the process X_t. Therefore, $u_t = u(t, \omega)$ is a stochastic process ((Øksendal [18])).

Optimal Control

In general, an optimal control problem consists of the following elements:

- State Process $Z(.)$: This process must capture of the minimal necessary information needed to describe the problem. $Z(t) \in \mathbb{R}^d$ is influenced by the control and given the control it has a Markovian structure.

- Control Process $\nu(.)$: The control set, U, in which $\nu(t)$ takes values in for every t needs to be described. In the stochastic setting, ν will be required to be adapted to a certain filtration to model the flow of information.

- Admissible controls $\mathcal{A}$: A control process satisfying the constraints is called an admissible control. The set of all admissible controls will be denoted by $\mathcal{A}$ and it may depend on the initial value of the state process.

- Objective functional $J\big(Z(.), \nu(.)\big)$: This is the functional to be maximized (or minimized).

Then ,the aim is to minimize (or minimize) the objective functional J over all admissible controls. The main problem in optimal control is to find the minimizing control. A partial differential equation is satisfied by the value function ν. Also, the optimal control in a "feedback" form is obtained. That is, the optimal process $\nu^*(t)$ is given as $\widehat{\nu}\,(Z^*(t))$, where $\widehat{\nu}$ is the optimal feedback control given as a function of the state and Z^* is the corresponding optimal state process (Soner [24]).

3.3   Dynamic Programming Principle

The Dynamic Programming Principle(DPP) is a fundamental principle in the theory of stochastic control (Pham [19]). The American scholar Berman et al put forward dynamic programming in 1951 which provides an effective approach to such issue as distribution of funds. The unique feature of dynamic programming is that it uses

decision-making by stages in the multi-variable complex decision-making issue, and changes it into a decision-making issue of solving several single variables (Yan [27]). This principle provides a general framework for analyzing many problem types. A variety of optimization techniques can be employed within this framework to solve particular aspects of a more general formulation. Mostly, creativity is required before we can recognize that a particular problem can be cast effectively as a dynamic program, and often subtle insights are necessary to restructure the formulation so that it can be solved effectively (Hajihassani [7]).

The concept of dynamic programming offers a unified approach to solving problems of stochastic control. Central to the methodology is the cost-to-go function, which can be obtained via solving Bellmann's equation. The domain of the cost-to-go function is the state space of the system to be controlled, and dynamic programming algorithms compute and store a table consisting of one cost-to-go value per state. Unfortunately, the size of a state space typically grows exponentially in the number of state variables. Known as the curse of dimensionality, this phenomenon renders dynamic programming untraceable in the face of problems of practical scale (Dietterich etal [25]).

The framework of controlled diffusion may be considered and the problem will be formulated on finite or infinite horizon. The basic idea of the approach is to consider a family of controlled problems by varying the initial state values, and to derive

some relations between the associated value functions. This approach yields a certain partial differential equation(PDE), of second order and non-linear , called Hamilton-Jacobi-Bellman(HJB). When this PDE can be solved by the explicit or theoretical achievement of a smooth solution, the verification theorem validates the optimality of the candidate solution to the HJB equation. This classical approach to the dynamic programming is called the Verification step. The main drawback of this approach is to suppose the existence of a regular solution to the HJB equation (Pham [19]).

The Hamilton Jacobi Bellman Euation (HJB) provides the globally optimal solution to large classes of control problems. Unfortunately, This generally comes out at a price, the calculation of such solutions is typically intractible for systems with more than moderate state space size due to the cause of dimensionality (Horowitz etal [10]).

Chapter 4

The Model

4.1   Introduction

With our solid base on the preliminary concepts which are discussed in the previous chapters, we now focus on the model.

The project is focused on finding the optimal investment under inflation protection and also to find out the optimal portfolio under stochastic wage income and stochastic cash outflows.

We will start by giving the general description of the model, and we will also discuss the dynamics of the relevant features which will be considered and these include; the dynamics of the wealth process, the discounted stochastic wage income process, the

discounted cash outflows process, the wealth valuation of the Investor, and finally the optimal portfolio strategies for the Investor.

4.2   Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}(t))_{0 \leq t \leq T}$, where

$\mathcal{F}_t = \sigma(W^I(t), W^S(t), W^w(t) : s \leq t)$, the brownian motions $W(t) = (W^I(t), W^S(t), W^w(t))'$

is a 3-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}(\mathcal{F}), \mathbb{P})$,

$t \in [0, T]$, where $\mathbb{P}$ is the real world probability measure, t the time period, T is the terminal time period.

$W^I(t)$ is the Brownian motion with respect to the source of uncertainty arising from inflation and $W^S(t)$ is the Brownian motion with respect to the source of uncertainty arising from stock market. $\sigma^S = (\sigma^S_1, \sigma^S_2)$ and $\sigma^I = (\sigma_I, 0)$ are the volatility vector of stock and volatility vector of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^I(t)$. $W^w(t)$ is a standard Brownian motion independent of $W^S(t)$ and $W^I(t)$. μ is the appreciation rate of stock. Moreover, σ^S and σ^I referred to as the co-officiants of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

We assume that the Investor faces a market that is characterized by a risk-free asset(cash account) and two risky assets, all of whom are tradeable. In this work, we

allow the stock price to be correlated to inflation. Also, we correlated the cash out-

flows to stock market in order to determine the extent to which the cash outflows

should be hedged. The dynamics of the underlying assets are given by (4.1):

$$
\begin{aligned}
dC(t) &= rC(t)dt \\
C(0) &= 1 \\
dS(t) &= \mu S(t)dt + \sigma_1^S S(t)dW^I(t) + \sigma_2^S S(t)dW^S(t) \\
S(0) &= s_o > 0 \qquad\qquad (4.1) \\
dB(t,Q(t)) &= (r + \sigma_I \theta^I)B(t,Q(t))dt + \sigma_I B(t,Q(t))dW^I(t) \\
B(0) &= b > 0
\end{aligned}
$$

where , r is the nominal interest rate,

θ^I is the price of inflation risk,

$C(t)$ is the price process of the cash account at time t,

$S(t)$ is stock price process at time t,

$Q(t)$ is the inflation index at time t and has the dynamics

$$dQ(t) = E(q)Q(t)dt + \sigma_I Q(t)dW^I(t)$$

where $E(q)$ is the expected rate of inflation, which is the difference between nominal

interest rate, r and real interest rate R (i.e. $E(q) = r - R$).

$B(t, Q(t))$ is the inflation-indexed bond price process at time t.

Then, the volatility matrix:

$$\Sigma = \begin{pmatrix} \sigma^I \\ \sigma^S \end{pmatrix} = \begin{pmatrix} \sigma_I & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix},$$

corresponding to the two risky assets and satisfies $det(\sum) = \sigma_I \sigma_2^S \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying;

$$\theta = \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \theta^I \\ \dfrac{\mu - r - \sigma_1^S \theta^I}{\sigma_2^S} \end{pmatrix}$$

where θ^S is the market price of stock risks and θ^I is the market price of inflation risks(MPIR).

θ was calculated as follows ,

$$\Sigma.\theta = \Upsilon,$$

where:

$$\Sigma = \begin{pmatrix} \sigma^I \\ \sigma^S \end{pmatrix} = \begin{pmatrix} \sigma_I & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix},$$

$$\theta = \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix}$$

and

$$\Upsilon = \begin{pmatrix} \sigma_I \theta^I \\ \mu - r \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \sigma_I & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix} \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \sigma_I \theta^I \\ \mu - r \end{pmatrix},$$

$$\sigma_I \theta^I = \sigma_I \theta^I \qquad (4.2)$$

$$\sigma_1^S \theta^I + \sigma_2^S \theta^S = \mu - r \qquad (4.3)$$

Therefore, by equation (4.2);

$$\theta^I = \theta^I$$

and

by equation(4.3);

$$\sigma_2^S \theta^S = \mu - r - \sigma_1^S \theta^I \tag{4.4}$$

Finally,

$$\theta^S = \frac{\mu - r - \sigma_1^S \theta^I}{\sigma_2^S} \tag{4.5}$$

With the assumption that the exponential process $Z(t)$ which we assumed to be Martingale in $\mathbb{P}$ to have the following Stochastic Differential Equation

$$dZ(t) = Z(t)\big(-\theta^I dW^I(t) - \theta^S dW^S(t) \big)$$

Solving for Z(t), we use Itô's formula for continuous processes.

Let $f(t) = \ln Z(t)$.

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial Z}dZ(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 Q}{\partial Z^2}(dZ(t))^2 \\
&= 0.dt + \frac{1}{Z(t)}dZ(t) + \frac{1}{2}\left(-\frac{1}{Z(t)^2}(dZ(t))^2\right) \\
&= \frac{1}{Z(t)}dZ(t) - \frac{1}{2}\frac{1}{Z(t)^2}(dZ(t))^2
\end{aligned}
$$

Calculating $[dZ(t)]^2$:

$$
\begin{aligned}
(dZ(t))^2 &= \left[Z(t)\left(-\theta^I dW^I(t) - \theta^S dW^S(t)\right)\right]^2 \\
&= \left(Z(t)(-\theta^I dW^I(t) - \theta^S dW^S(t))\right).\left(Z(t)(-\theta^I dW^I(t) - \theta^S dW^S(t))\right) \\
&= \left(-Z(t)\theta^I dW^I(t) - Z(t)\theta^S dW^S(t)\right).\left(-Z(t)\theta^I dW^I(t) - Z(t)\theta^S dW^S(t)\right)
\end{aligned}
$$

But,

$$
dt.dt = dt.dW^I(t) = dt.dW^S(t) = dW^I(t).dt = 0
$$

$$
dW^I(t).dW^S(t) = dW^S(t).dt = dW^S(t).dW^I(t) = 0
$$

$$
dW^S(t).dW^S = dW^I(t).dW^I(t) = dt
$$

Now,

$$
\begin{aligned}
(dZ(t))^2 &= Z^2(\theta^I)^2 dt + Z^2(\theta^S)^2 dt \\
&= Z^2(t)\Big((\theta^I)^2 + (\theta^S)^2\Big)dt
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
df &= \frac{1}{Z(t)}dZ(t) - \frac{1}{2}\frac{1}{Z(t)^2}(dZ(t))^2 \\
&= \frac{1}{Z(t)}[Z(t)(-\theta^I dW^I(t) - \theta^S dW^S(t)] - \frac{1}{2}\frac{1}{Z(t)^2}\Big[Z^2(t)\big((\theta^I)^2 + (\theta^S)^2\big)dt\Big] \\
&= -\theta^I dW^I(t) - \theta^S dW^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)dt
\end{aligned}
$$

But, $f(t) = \ln Z(t)$.

Therefore,

$$
d\big(\ln Z(t)\big) = -\theta^I dW^I(t) - \theta^S dW^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)dt \qquad (4.6)
$$

Taking integral both sides of (4.6):

$$\int_0^t d\big(\ln Z(p)\big) = \int_0^t \Big[-\theta^I dW^I(p) - \theta^S dW^S(p) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)dp \Big]$$

$$\ln Z(t) - \ln Z(0) = -\theta^I W^I(t) + \theta^I W^I(0) - \theta^S W^S(t) + \theta^S W^S(0)$$

$$-\frac{1}{2}\Big((\theta^I)^2 + (\theta^S)^2\Big)t - \frac{1}{2}\Big((\theta^I)^2 + (\theta^S)^2\Big).0$$

$W(0) = (W^I(0), W^S(0), W^w(0))' = (0,0,0)'$,then we have:

$$\ln Z(t) - \ln Z(0) = -\theta^I W^I(t) - \theta^S W^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)t$$

$$\ln Z(t) = lnZ(0) - \theta^I W^I(t) - \theta^S W^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)t \quad (4.7)$$

Taking the exponential both sides of the equation (4.7), we get:

$$\exp\Big[\ln Z(t)\Big] = \exp\Big[lnZ(0) - \theta^I W^I(t) - \theta^S W^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)t\Big]$$

$$Z(t) = Z(0)\exp\Big[-\theta^I W^I(t) - \theta^S W^S(t) - \frac{1}{2}\big((\theta^I)^2 + (\theta^S)^2\big)t\Big]$$

Let $Z(0) = 1$,

$$Z(t) = \exp\left[-\theta^I W^I(t) - \theta^S W^S(t) - \frac{1}{2}\left((\theta^I)^2 + (\theta^S)^2\right)t\right]$$

We assume an investor gives a stochastic income over the time t, and the income rate at time t is $a(Y_t, t)$. Y_t is the state variable.

$$dY_t = \alpha(Y_t, t)dt + \beta(Y_t, t)dV(t) \tag{4.8}$$

The correlation between $dW^S(t)$ and $dV(t)$ is ρ dt where $\rho \in [-1, 1]$.

V_t can be written in the form $V_t = \rho W^S(t) + \sqrt{(1 - \rho^2)}W^w(t)$ where $W^w(t)$ is a standard Brownian Motion independent of $W^S(t)$.

Also, we assume cash outflows process $L(t)$ at time t follows the dynamic:

$$
\begin{aligned}
dL(t) &= L(t)\left(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)\right), \tag{4.9}\\
L(0) &= L_0 > 0
\end{aligned}
$$

where, $\delta > 0$ is the expected growth rate of the cash outflows and σ_1^L is the volatility caused by the source of inflation, $W^I(t)$ and σ_2^L is the volatility caused by the source of uncertainty arises from the stock market, $W^S(t)$.

4.2.1 The Dynamics of the Stochastic Wage Income

Solving for Y(t) we use Itô's lemma for continuous processes and apply it on (4.8).

We take a special case where we let $\alpha(Y_t, t) = \alpha Y_t$ and $\beta(Y_t, t) = \beta Y_t$.

Let $f - C^{1,2}$ and consider the stochastic wage income rate $a(Y_t, t) = f(y, t)$.

We know, $V_t = \rho W^S(t) + \sqrt{(1 - \rho^2)} W^w(t)$.

Therefore, $dV_t = \rho dW^S(t) + \sqrt{(1 - \rho^2)} dW^w(t)$

This implies,

$$
\begin{aligned}
dY_t &= \alpha Y_t dt + \beta Y_t dV_t \\
dY_t &= \alpha Y_t dt + \beta Y_t \left[\rho dW^S(t) + \sqrt{(1 - \rho^2)} dW^w(t) \right] \\
dY_t &= Y(T) \left(\alpha dt + \beta \left[\rho dW^S(t) + \sqrt{(1 - \rho^2)} dW^w(t) \right] \right) \\
dY_t &= Y(T) \left(\alpha dt + \beta \rho dW^S(t) + \beta \sqrt{(1 - \rho^2)} dW^w(t) \right)
\end{aligned}
$$

Let $f(t) = \ln Y(t)$.

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial Y}dY(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2}(dY(t))^2 \\
&= 0.dt + \frac{1}{Y(t)}dL(t) + \frac{1}{2}\left(-\frac{1}{Y(t)^2}\big(dY(t)\big)^2\right) \\
&= \frac{1}{Y(t)}dY(t) - \frac{1}{2}\frac{1}{Y(t)^2}\big(dY(t)\big)^2
\end{aligned}
$$

Calculating $[dY(t)]^2$:

$$
\begin{aligned}
(dY(t))^2 &= \left[Y(t)\big(\alpha dt + \beta\rho dW^S(t) + \beta\sqrt{(1-\rho^2)}dW^w(t)\big)\right]^2 \\
&= \left(Y(t)\big(\alpha dt + \beta\rho dW^S(t) + \beta\sqrt{(1-\rho^2)}dW^w(t)\big)\right) \\
&\quad .\left(Y(t)\big(\alpha dt + \beta\rho dW^S(t) + \beta\sqrt{(1-\rho^2)}dW^w(t)\big)\right)
\end{aligned}
$$

$$
\begin{aligned}
(dY(t))^2 &= Y^2(t)\Big[\alpha^2(dt.dt) + \alpha\beta\rho(dt.dW^S(t)) + \alpha\beta\sqrt{(1-\rho^2)}(dt.dW^w(t)) \\
&\quad + \beta\rho\alpha(dW^S(t).dt) + \beta^2\rho^2(dW^S(t).dW^S(t)) + \beta^2\rho\sqrt{(1-\rho^2)}(dW^S(t).dW^w(t)) \\
&\quad + \beta\sqrt{(1-\rho^2)}\alpha(dW^w(t).dt) + \beta\sqrt{(1-\rho^2)}\beta\rho(dW^w(t).dW^S(t)) \\
&\quad + \beta^2(\sqrt{(1-\rho^2)})^2(dW^w(t).dW^w(t))\Big]
\end{aligned}
$$

But,

$$dt.dt = dt.dW^I(t) = dt.dW^S(t) = dW^I(t).dt = 0$$

$$dW^I(t).dW^S(t) = dW^S(t).dt = dW^S(t).dW^I(t) = 0$$

$$dW^w(t).dW^S(t) = dW^S(t).dW^w = dW^S(t).dW^w(t) = 0$$

$$dW^w(t).dW^S(t) = dt.dW^w(t) = dW^w(t).dt = 0$$

$$dW^S(t).dW^S = dW^I(t).dW^I(t) = dW^w(t).dW^w(t) = dt$$

Now,

$$(dY(t))^2 = Y^2(t)\left[\beta^2\rho^2 dt + \beta^2(1-\rho^2)dt\right] \qquad (4.10)$$

Therefore,

$$
\begin{aligned}
df &= \frac{1}{Y(t)}dY(t) - \frac{1}{2}\frac{1}{Y(t)^2}\left(dY(t)\right)^2 \\
&= \frac{1}{Y(t)}dY(t) - \frac{1}{2}\frac{1}{Y^2(t)}\left[Y^2(t)\left(\beta^2\rho^2 dt + \beta^2(1-\rho^2)dt\right)\right] \\
&= \frac{1}{Y(t)}dY(t) - \frac{1}{2}[\beta^2\rho^2 dt + (1-\rho^2)dt] \\
&= \frac{1}{Y(t)}\left[Y(t)\left(\alpha dt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)\right)\right] - \frac{1}{2}\left(\beta^2\rho^2 dt + \beta^2(1-\rho^2)dt\right) \\
&= \alpha dt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t) - \frac{1}{2}\left(\beta^2\rho^2 dt + \beta^2(1-\rho^2)dt\right) \\
&= \alpha dt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t) - \frac{1}{2}\beta^2\rho^2 dt - \frac{1}{2}\beta^2(1-\rho^2)dt \\
&= \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)dt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)
\end{aligned}
$$

But, $f(t) = \ln Y(t)$.

Therefore,

$$
\begin{aligned}
d(\ln Y(t)) &= \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)dt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t) \\
\int_0^t d[\ln Y(p)] &= \int_0^t \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)dp \\
&\quad + \int_0^t \beta\rho dW^S(p) + \int_0^t \beta\sqrt{1-\rho^2}dW^w(p) \\
\ln Y(t) - \ln Y(0) &= \int_0^t \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)dp \\
&\quad + \int_0^t \beta\rho dW^S(p) + \int_0^t \beta\sqrt{1-\rho^2}dW^w(p) \\
\ln Y(t) &= \ln Y(0) + \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)t - \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2\sqrt{1-\rho^2}\right).0 \\
&\quad + \beta\rho W^S(t) - \beta\rho dW^S(0) + \beta\sqrt{1-\rho^2}dW^w(t) - \beta\sqrt{1-\rho^2}dW^w(0)
\end{aligned}
$$

$W(0) = (W^I(0), W^S(0), W^w(0))' = (0,0,0)'$, then:

$$
\ln Y(t) = \ln Y(0) + \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)t + \beta\rho W^S(t) + \beta\sqrt{1-\rho^2}W^w(t)
$$

Taking the exponential both sides of the equation:

$$\exp\left[\ln Y(t)\right] = \exp\left[\ln Y(0) + \left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)t + \beta\rho W^S(t) + \beta\sqrt{1-\rho^2}W^w(t)\right]$$

$$Y(t) = Y(0)\exp\left[\left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)t + \beta\rho W^S(t) + \beta\sqrt{1-\rho^2}W^w(t)\right]$$

But $Y(0) = Y_0 > 0$,

$$Y(t) = Y_0\exp\left[\left(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2)\right)t + \beta\rho W^S(t) + \beta\sqrt{1-\rho^2}W^w(t)\right]$$

4.2.2   The Dynamics of the Stochastic Cash Outflows

We now apply Itô lemma on (4.9);

Solving for $L(t)$, we use Itô's formula for continuous processes.

Let $f(t) = \ln L(t)$.

$$
\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial L}dL(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial L^2}\left(dL(t)\right)^2 \\
&= 0.dt + \frac{1}{L(t)}dL(t) + \frac{1}{2}\left(-\frac{1}{L(t)^2}\left(dL(t)\right)^2\right) \\
&= \frac{1}{L(t)}dL(t) - \frac{1}{2}\frac{1}{L(t)^2}\left(dL(t)\right)^2
\end{aligned}
$$

Calculating $[dL(t)]^2$:

$$
\begin{aligned}
(dL(t))^2 &= \left[L(t)\big(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)\big) \right]^2 \\
&= \left(L(t)\big(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)\big) \right) \\
&\quad \cdot \left(L(t)\big(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)\big) \right)
\end{aligned}
$$

$$
\begin{aligned}
(dL(t))^2 = L^2(t)\Big[&\delta^2(dt).dt + \delta\sigma_1^L dt.dW^I(t) + \delta\sigma_2^L dt.dW^S(t) \\
&+ \sigma_1^L \delta dW^I(t).dt + \sigma_1^L \sigma_1^L dW^I(t).dW^I(t) + \sigma_1^L \sigma_2^L dW^I(t).dW^S(t) \\
&+ \sigma_2^L \delta dW^S(t).dt + \sigma_2^L \sigma_1^L dW^S(t)dW^I(t) + \sigma_2^L \sigma_2^L dW^S(t).dW^S(t) \Big]
\end{aligned}
$$

But,

$$
dt.dt = dt.dW^I(t) = dt.dW^S(t) = dW^I(t).dt = 0
$$

$$
dW^I(t).dW^S(t) = dW^S(t).dt = dW^S(t).dW^I(t) = 0
$$

$$
dW^S(t).dW^S = dW^I(t).dW^I(t) = dt
$$

Now,

$$(dL(t))^2 = L^2(t)[(\sigma_1^L)^2 dt + (\sigma_2^L)^2 dt] \tag{4.11}$$

Therefore,

$$
\begin{aligned}
df &= \frac{1}{L(t)} dL(t) - \frac{1}{2} \frac{1}{L(t)^2} (dL(t))^2 \\
&= \frac{1}{L(t)} dL(t) - \frac{1}{2} \frac{1}{L(t)^2} \left[L^2(t) \left((\sigma_1^L)^2 dt + (\sigma_2^L)^2 dt \right) \right] \\
&= \frac{1}{L(t)} dL(t) - \frac{1}{2} [(\sigma_1^L)^2 dt + (\sigma_2^L)^2 dt] \\
&= \frac{1}{L(t)} \left[L(t).\left(\delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t) \right) \right] - \frac{1}{2} \left((\sigma_1^L)^2 dt + (\sigma_2^L)^2 dt \right) \\
&= \delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t) - \frac{1}{2} \left((\sigma_1^L)^2 dt + (\sigma_2^L)^2 dt \right) \\
&= \delta dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t) - \frac{1}{2} (\sigma_1^L)^2 dt - \frac{1}{2} (\sigma_2^L)^2 dt \\
&= \left(\delta - \frac{1}{2} (\sigma_1^L)^2 - \frac{1}{2} (\sigma_2^L)^2 \right) dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)
\end{aligned}
$$

But, $f(t) = \ln L(t)$.

Therefore,

$$
\begin{aligned}
d(\ln L(t)) &= \left(\delta - \frac{1}{2} (\sigma_1^L)^2 - \frac{1}{2} (\sigma_2^L)^2 \right) dt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t) \\
\int_0^t d[\ln L(p)] &= \int_0^t \left(\delta - \frac{1}{2} \sigma_1^L - \frac{1}{2} \sigma_2^L \right) dp + \int_0^t \sigma_1^L dW^I(p) + \int_0^t \sigma_2^L dW^S(p) \\
\ln L(t) - \ln L(0) &= \int_0^t \left(\delta - \frac{1}{2} (\sigma_1^L)^2 - \frac{1}{2} (\sigma_2^L)^2 \right) dp \int_0^t \sigma_1^L dW^I(p) + \int_0^t \sigma_2^L dW^S(p)
\end{aligned}
$$

$$\ln L(t) \;=\; \ln L(0) + \left(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2\right)t - \left(\delta - \frac{1}{2}\frac{1}{L(t)^2}\right).0$$

$$+\sigma_1^L W^I(t) - \sigma_1^L W^I(0) + \sigma_2^L W^S(t) - \sigma_2^L W^S(0)$$

$W(0) = (W^I(0), W^S(0), W^w(0))' = (0,0,0)'$, then:

$$\ln L(t) = \; lnL(0) + \left(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2\right)t + \sigma_1^L W^I(t) + \sigma_2^L W^S(t) \qquad (4.12)$$

Taking the exponential both sides of equation (4.12):

$$\exp\left[\ln L(t)\right] \;=\; \exp\left[\ln L(0) + \left(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2\right)t + \sigma_1^L W^I(t) + \sigma_2^L W^S(t)\right]$$

$$L(t) \;=\; L(0)\exp\left[\left(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2\right)t + \sigma_1^L W^I(t) + \sigma_2^L W^S(t)\right]$$

But $L(0) = L_0 > 0$,

$$L(t) = L_0\exp\left[\left(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2\right)t + \sigma_1^L W^I(t) + \sigma_2^L W^S(t)\right]$$

4.3   The Dynamics of the Wealth Process

Let $X^{\Delta,Y,L}(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in the stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t.

We then define the corresponding continuous and adapted wealth process $X^{\Delta,Y,L}(t), t \in [0,T]$ with respect to the self-financing trading strategy Δ as:

$$
\begin{aligned}
dX^{\Delta,Y,L}(t) &= \Delta^S(t)X^{\Delta,Y,L}(t)\frac{dS(t)}{S(t)} \\
&+ \Delta^I(t)X^{\Delta,Y,L}(t)\frac{dB(t,Q(t))}{B(t,Q(t))} \\
&+ \big(1 - \Delta^S(t) - \Delta^I(t)\big)X^{\Delta,Y,L}(t)\frac{dC(t)}{C(t)} \\
&+ \big(Y(t) - L(t)\big)dt, \\
&= \Delta^S(t)X^{\Delta,Y,L}(t)\big[\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t)\big] \\
&+ \Delta^I(t)X^{\Delta,Y,L}(t)\big[(r + \sigma_I\theta^I)dt + \sigma_I dW^I(t)\big] \\
&+ \big(1 - \Delta^S(t) - \Delta^I(t)\big)X^{\Delta,Y,L}(t)\big[rdt\big] + \big(Y(t) - L(t)\big)dt \quad (4.13)
\end{aligned}
$$

4.4   Discounted Income and Discounted Outflow Processes

Definition 4.4.1. The expected discounted stochastic wage income (EDSWI) process at time t is defined as:

$$\Psi(t) = \mathbb{E}\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} Y(u) du | \mathcal{F}(t)\right], T \geq t, \tag{4.14}$$

where $\Lambda(t) = \dfrac{Z(t)}{C(t)} = \exp(-rt)Z(t)$ is the stochastic discount factor which adjusts for nominal interest rate and market price of risks, ad $\mathbb{E}(.|\mathcal{F}(t))$ is a real world conditional expectation with respect to the Brownian filtration $(\mathcal{F}(t))_{t\geq 0}$.

4.4.1   Proposition 1

Let $\Psi(t)$ be the expected discounted stochastic wage income process, then:

$$\Psi(t) = \frac{Y(t)}{\phi}\left[\exp(\phi(T-t)-1)\right], \text{ where } \phi = \alpha - r - (\theta^I)^2 - (\theta^S)^2$$

Proof. By definition:

$$
\begin{aligned}
\Psi(t) &= \mathbb{E}\left[\int_t^T \frac{\Lambda(u)}{\Lambda(t)} Y(u) du | \mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\int_t^T \frac{\Lambda(u)Y(t)}{\Lambda(t)Y(t)} Y(u) du | \mathcal{F}(t)\right] \\
&= \varphi(t)\mathbb{E}\left[\int_t^T \frac{\Lambda(u)Y(u)}{\Lambda(t)Y(t)} du | \mathcal{F}(t)\right]
\end{aligned}
$$

But the processes $\Lambda(.)$ and $\varphi(.)$ are geometric Brownian motions and it follows that $\dfrac{\Lambda(u)Y(u)}{\Lambda(t)Y(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Adopting change of variables, we have,

$$
\text{Let } n = u - t, u \geq t,
$$

$$
\begin{aligned}
\Psi(t) &= Y(t)\mathbb{E}\left[\int_t^T \frac{\Lambda(u)Y(u)}{\Lambda(t)Y(t)} du | \mathcal{F}(t)\right] \\
&= Y(t)\mathbb{E}\left[\int_{t-t}^{T-t} \frac{\Lambda(u-t)Y(u-t)}{\Lambda(t-t)Y(t-t)} d(u-t) | \mathcal{F}(t-t)\right] \\
&= Y(t)\mathbb{E}\left[\int_0^{T-t} \frac{\Lambda(n)Y(n)}{\Lambda(0)Y(0)} dn | \mathcal{F}(0)\right]
\end{aligned}
$$

But,

$$
\begin{aligned}
\Lambda(0) &= \frac{Z(0)}{C(0)} \\
&= \exp(-r(0))Z(0) \\
&= \exp(0)Z(0) \\
&= 1
\end{aligned}
$$

Therefore,

$$\Psi(t) \;=\; Y(t)\mathbb{E}\left[\int_0^{T-t}\frac{\Lambda(n)Y(n)}{Y(0)}dn|\mathcal{F}(0)\right]$$

Now,

$$
\begin{aligned}
\frac{\Lambda(n)Y(n)}{Y(0)} &= \Lambda(n)\frac{Y(n)}{Y(0)} \\[2mm]
&= \exp(-rn)Z(n)\frac{Y(n)}{Y(0)} \\[2mm]
&= \exp(-rn)\exp(-\theta^I W^I(n) - \theta^S W^S(n) - \frac{1}{2}((\theta^I)^2 + (\theta^S)^2)n) \\[2mm]
&\quad . \frac{Y(0)\exp\left[(\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2))n + \beta\rho W^S(n) + \beta\sqrt{1-\rho^2}W^w(n)\right]}{Y(0)} \\[2mm]
&= \exp\left(-rn\right)\exp\left(-\theta^I W^I(n) - \theta^S W^S(n) - \frac{1}{2}((\theta^I)^2 + (\theta^S)^2)n\right) \\[2mm]
&\quad . \exp\left((\alpha - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}(1-\rho^2))n + \beta\rho W^S(n) + \beta\sqrt{1-\rho^2}W^w(n)\right) \quad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Lambda(n)Y(n)}{Y(0)} &= \exp\left(-rn\right)\exp\left(-\theta^I W^I(n) - \theta^S W^S(n) - \frac{1}{2}(\theta^I)^2 n - \frac{1}{2}(\theta^S)^2 n\right) \\[2mm]
&\quad . \exp\left(\alpha n - \frac{1}{2}\beta^2\rho^2 n - \frac{1}{2}\beta^2(1-\rho^2)n + \beta\rho W^S(n) + \beta\sqrt{1-\rho^2}W^w(n)\right) \\[2mm]
&= \exp\left[(\alpha - r)n - \frac{1}{2}((\theta^I)^2 + (\theta^S)^2 + \beta^2\rho^2 + \beta^2(1-\rho^2))n\right] \\[2mm]
&\quad . \exp\left(-\theta^I W^I(n) - \theta^S W^S(n) + \beta\rho W^S(n) + \beta\sqrt{1-\rho^2}W^w(n)\right)
\end{aligned}
$$

Finally,

$$\Psi(t) \;=\; Y(t)\mathbb{E}\left[\int_0^{T-t}\exp\left[(\alpha-r)n-\frac{1}{2}\big((\theta^I)^2+(\theta^S)^2+\beta^2\rho^2+\beta^2(1-\rho^2)\big)n\right]\right.$$
$$\left.\cdot\exp\left(-\theta^I W^I(n)-\theta^S W^S(n)+\beta\rho W^S(n)+\beta\sqrt{1-\rho^2}W^w(n)\right)\right]dn$$
$$=\; Y(t)\int_0^{T-t}\mathbb{E}\left[\exp\big((\alpha-r)n\big)\right]\mathbb{E}\left[\exp\left(-\frac{1}{2}\big((\theta^I)^2+(\theta^S)^2+\beta^2\rho^2+\beta^2(1-\rho^2)\big)n\right)\right]$$
$$\cdot\mathbb{E}\left[\exp\left(-\theta^I W^I(n)-\theta^S W^S(n)+\beta\rho W^S(n)+\beta\sqrt{1-\rho^2}W^w(n)\right)\right]dn$$

Assuming that the Brownian motions are independent, we have:

$$\mathbb{E}\left[\exp(\beta\rho^2 W^S(n))\right] \;=\; \exp\left(\frac{1}{2}\beta^2\rho^2 n\right)$$

$$\mathbb{E}\left[\exp(\beta\sqrt{1-\rho^2}W^S(n))\right] \;=\; \exp\left(\frac{1}{2}\beta^2(1-\rho^2)n\right)$$

$$\mathbb{E}\left[\exp(\theta^I W^I(n))\right] \;=\; \exp\left(\frac{1}{2}(\theta^I)^2 n\right)$$

$$\mathbb{E}\left[\exp(\theta^S W^S(n))\right] \;=\; \exp\left(\frac{1}{2}(\theta^S)^2 n\right)$$

This implies that:

$$\Psi(t) \;=\; Y(t)\int_0^{T-t}\left[\exp\big((\alpha-r)n\big)\right]\left[\exp\left(-\frac{1}{2}\big((\theta^I)^2+(\theta^S)^2+\beta^2\rho^2+\beta^2(1-\rho^2)\big)n\right)\right]$$

$$\cdot \left[\exp\left(-\frac{1}{2}(\theta^I)^2 n - \frac{1}{2}(\theta^S)^2 n + \frac{1}{2}\beta^2\rho^2 n + \frac{1}{2}\beta^2(1-\rho^2)n \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \left[\exp\left((\alpha-r)n\right) \right] \left[\exp\left(\left(-\frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 - \frac{1}{2}\beta^2\rho^2 - \frac{1}{2}\beta^2(1-\rho^2) \right)n \right) \right]$$

$$\cdot \left[\exp\left(-\frac{1}{2}(\theta^I)^2 n - \frac{1}{2}(\theta^S)^2 n + \frac{1}{2}\beta^2\rho^2 n + \frac{1}{2}\beta^2(1-\rho^2)n \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \left[\exp\left((\alpha-r)n\right) \right] \left[\exp\left(\left(-\frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 \right)n \right) \right]$$

$$\cdot \left[\exp\left(\left(-\frac{1}{2}(\theta^I)^2 n - \frac{1}{2}(\theta^S)^2 n \right) \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \left[\exp\left((\alpha-r)n\right) \right] \left[\exp\left(\left(-\frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 - \frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 \right)n \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \left[\exp\left((\alpha-r)n\right) \right] \left[\exp\left(\left(-(\theta^I)^2 - (\theta^S)^2 \right)n \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \left[\exp\left(\left(\alpha - r - (\theta^I)^2 - (\theta^S)^2 \right)n \right) \right] dn$$

$$= Y(t) \int_0^{T-t} \exp\left(\left(\alpha - r - (\theta^I)^2 - (\theta^S)^2 \right)n \right) dn$$

$$= Y(t) \int_0^{T-t} \exp(\phi n) dn$$

where $\phi = \alpha - r - (\theta^I)^2 - (\theta^S)^2$.

Therefore,

$$\Psi(t) = \frac{(Y(t)}{\phi} \left[\exp\left(\phi(T-t) - 1 \right) \right] \tag{4.16}$$

By Proposition 1, the value of expected future stochastic wage income process $\Psi(t)$

is proportional to the instantaneous total stochastic wage income process $Y(t)$.

Taking the differential of both sides of 4.16 , we obtain;

$$
\begin{aligned}
d\Psi(t) &= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}dY(t) \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}\left[Y(t)\left(\alpha dt + \beta\rho dW^S(t) + \beta\sqrt{(1-\rho^2)}dW^w(t)\right)\right] \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\alpha dt + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\rho dW^S(t) \\
&\quad + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\sqrt{1-\rho^2}dW^w(t) \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\alpha dt + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\rho dW^S(t) \\
&\quad + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\sqrt{1-\rho^2}dW^w(t) + Y(t)dt - Y(t)dt \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\alpha dt + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\frac{\phi}{\exp\left(\phi(T-t)-1\right)}dt \\
&\quad + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\rho dW^S(t) \\
&\quad + \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\beta\sqrt{1-\rho^2}dW^w(t) - Y(t)dt \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\left[\left(\alpha + \frac{\phi}{exp\left(\phi(T-t)-1\right)}\right)dt \right. \\
&\quad \left. + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)\right] - Y(t)dt \\
&= \frac{\exp\left(\phi(T-t)-1\right)}{\phi}Y(t)\left[Fdt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)\right] - Y(t)dt
\end{aligned}
$$

Therefore

$$
\begin{aligned}
d\Psi(t) &= \Psi(t)\left[Fdt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)\right] - Y(t)dt \quad (4.17) \\
\text{where } F &= \frac{\alpha\left[\exp\left(\phi(T-t)-1\right)\right] + \phi}{\exp\left(\phi(T-t)-1\right)}
\end{aligned}
$$

□

Definition 4.4.2. The expected discounted cash outflows process at time t is defined as:

$$\Phi(t) = \mathbb{E}[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)} L(u)du|\mathcal{F}(t)], T \geq t), \tag{4.18}$$

where $\Lambda(t) = \dfrac{Z(t)}{C(t)} = exp(-rt)Z(t)$ is the stochastic discount factor which adjusts for nominal interest rate and market price of risks, ad $\mathbb{E}(.|\mathcal{F}(t))$ is a real world conditional expectation with respect to the Brownian filtration $(\mathcal{F}(t))_{t\geq 0}$.

4.4.2   Proposition 2

Let Φ(t) be the expected discounted cash outflows(EDCO) process, then:

$\Phi(t) = \dfrac{L(t)}{\eta}\big(\exp(\eta T) - 1\big)$, where $\eta = \delta - r - (\theta^I)^2 - (\theta^S)^2$

Proof. By definition:

$$\begin{aligned}
\Phi(t) &= \mathbb{E}\left[\int_t^{T+t} \frac{\Lambda(u)}{\Lambda(t)} L(u)du|\mathcal{F}(t)\right] \\
&= \mathbb{E}\left[\int_t^{T+t} \frac{\Lambda(u)L(t)}{\Lambda(t)L(t)} L(u)du|\mathcal{F}(t)\right] \\
&= L(t)\mathbb{E}\left[\int_t^{T+t} \frac{\Lambda(u)L(u)}{\Lambda(t)L(t)} du|\mathcal{F}(t)\right]
\end{aligned}$$

But the process $\Lambda(.)$ and $L(.)$ are geometric Brownian motions and it follows that

$\dfrac{\Lambda(u)L(u)}{\Lambda(t)L(t)}$ is independent of the Brownian filtration $\mathcal{F}(t)$, $u \geq t$. Adopting change of

variables, we have,

$$
\begin{aligned}
\text{Let } \tau &= u - t, u \geq t, \\
\Phi(t) &= L(t)\mathbb{E}\left[\int_t^{T+t} \frac{\Lambda(u)L(u)}{\Lambda(t)L(t)}du|\mathcal{F}(t)\right] \\
&= L(t)\mathbb{E}\left[\int_{t-t}^{T+t-t} \frac{\Lambda(u-t)L(u-t)}{\Lambda(t-t)L(t-t)}d(u-t)|\mathcal{F}(t-t)\right] \\
&= L(t)\mathbb{E}\left[\int_0^T \frac{\Lambda(\tau)L(\tau)}{\Lambda(0)L(0)}d(\tau)|\mathcal{F}(0)\right] \\
\text{But, } \Lambda(0) &= \frac{Z(0)}{C(0)} \\
&= \exp(-r(0))Z(0) \\
&= \exp(0)Z(0) \\
&= 1
\end{aligned}
$$

Therefore

$$
\Phi(t) = L(t)\mathbb{E}\left[\int_0^T \frac{\Lambda(\tau)L(\tau)}{L(0)}d(\tau)|\mathcal{F}(0)\right]
$$

Now,

$$
\frac{\Lambda(\tau)L(\tau)}{L(0)} = \Lambda(\tau)\frac{L(\tau)}{L(0)}
$$

$$
\begin{aligned}
&= \exp(-r\tau)Z(\tau)\frac{L(\tau)}{L(0)} \\
&= \exp(-r\tau)Z(\tau)\frac{L(0)\exp[(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)]}{L(0)} \\
&= \exp(-r\tau)Z(\tau)\exp[(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)] \\
&= \exp(-r\tau)\exp(-\theta^I W^I(\tau) - \theta^S W^S(\tau) - \frac{1}{2}(\theta^I)^2\tau - \frac{1}{2}(\theta^S)^2\tau) \\
&\quad . \exp[(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)] \qquad (4.19)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\Phi(t) &= L(t)\mathbb{E}[\int_0^T exp(-r\tau)exp(-\theta^I W^I(\tau) - \theta^S W^S(\tau) - \frac{1}{2}(\theta^I)^2\tau - \frac{1}{2}(\theta^S)^2\tau) \\
&\quad . \exp[(\delta - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)] \\
&= L(t)\mathbb{E}[\int_0^T \exp[((\delta - r) - \frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau] \\
&\quad . \exp[-\theta^I W^I(\tau) - \theta^S W^S(\tau) + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)]
\end{aligned}
$$

Assuming the brownian motions are independent, we have:

$$
\begin{aligned}
\Phi(t) &= L(t)\int_0^T \mathbb{E}[\exp[((\delta - r) - \frac{1}{2}(\theta^I)^2 - \frac{1}{2}(\theta^S)^2 - \frac{1}{2}(\sigma_1^L)^2 - \frac{1}{2}(\sigma_2^L)^2)\tau] \\
&\quad . \mathbb{E}[\exp[-\theta^I W^I(\tau) - \theta^S W^S(\tau) + \sigma_1^L W^I(\tau) + \sigma_2^L W^S(\tau)]]
\end{aligned}
$$

But,

$$\mathbb{E}\big[\exp(\theta^I W^I(\tau))\big] = \exp\Big(\frac{1}{2}(\theta^I)^2\tau\Big)$$

$$\mathbb{E}\big[\exp(\theta^I W^S(\tau))\big] = \exp\Big(\frac{1}{2}(\theta^S)^2\tau\Big)$$

$$\mathbb{E}\big[\exp(\sigma_1^L W^I(\tau))\big] = \exp\Big(\frac{1}{2}(\sigma_1^L)^2\tau\Big)$$

$$\mathbb{E}\big[\exp(\sigma_2^L W^S(\tau))\big] = \exp\Big(\frac{1}{2}(\sigma_2^L)^2\tau\Big)$$

This implies that

$$
\begin{aligned}
\Phi(t) &= L(t)\int_0^T \exp[((\delta-r)-\frac{1}{2}(\theta^I)^2-\frac{1}{2}(\theta^S)^2-\frac{1}{2}(\sigma_1^L)^2-\frac{1}{2}(\sigma_2^L)^2)\tau]\\
&\quad .\exp[-\frac{1}{2}(\theta^I)^2\tau-\frac{1}{2}(\theta^S)^2\tau+\frac{1}{2}(\sigma_1^L)^2\tau+\frac{1}{2}(\sigma_2^L)^2\tau]
\end{aligned}
$$

$$
\begin{aligned}
\Phi(t) &= L(t)\int_0^T \exp[((\delta-r-\frac{1}{2}(\theta^I)^2-\frac{1}{2}(\theta^S)^2)\tau]\\
&\quad .\exp[-\frac{1}{2}(\theta^I)^2\tau-\frac{1}{2}(\theta^S)^2\tau]\\
&= L(t)\int_0^T \exp[((\delta-r-\frac{1}{2}(\theta^I)^2-\frac{1}{2}(\theta^S)^2-\frac{1}{2}(\theta^I)^2\tau-\frac{1}{2}(\theta^S)^2)\tau]
\end{aligned}
$$

$$= L(t) \int_0^T \exp[((\delta - r - (\theta^I)^2 - (\theta^S)^2)\tau]$$

$$= L(t) \int_0^T \exp(\eta\tau)d\tau$$

where $\eta = \delta - r - (\theta^I)^2 - (\theta^S)^2$.

Therefore,

$$\Phi(t) = \frac{L(t)}{\eta}[\exp(\eta T) - 1)] \tag{4.20}$$

Proposition 2 tells us that the expected discounted cash process $\Phi(t)$ is proportional to the instantaneous total cash outflows process $L(t)$.

□

Definition 4.4.3. Let $\Phi(t)$ be the expected discounted cash outflows process, then:

$$d\Phi(t) = \Phi(t)[qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)] - L(t)dt$$

$$\text{where} \quad q = \frac{\delta(\exp(\eta T) - 1) + \eta}{\exp(\eta T) - 1} \tag{4.21}$$

Proof. Taking the differential of both sides of (4.20), we obtain:

$$
\begin{aligned}
d\Phi(t) \;=\; & \frac{exp(\eta T) - 1}{\eta} dL(t) \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} \big[L(t)(\delta dt + \sigma_1^L dW^I(t) + \sigma_2 L dW^S(t))\big] \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} L(t)\delta dt + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_1^L dW^I(t) \\[2mm]
& + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_2 L dW^S(t) \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} L(t)\delta dt + L(t)dt + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_1^L dW^I(t) \\[2mm]
& + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_2 L dW^S(t) - L(t)dt \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} L(t)\delta dt + \Big(\frac{\exp(\eta T) - 1}{\eta}\Big)L(t)\frac{\eta}{\exp(\eta T) - 1}dt \\[2mm]
& + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_1^L dW^I(t) + \frac{\exp(\eta T) - 1}{\eta} L(t)\sigma_2^L dW^S(t) - L(t)dt \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} L(t)\big[(\delta + \frac{\eta}{\exp(\eta T) - 1})dt + \sigma_1^d W^I(t) + \sigma_2^L dW^S(t)\big] - L(t)dt \\[2mm]
=\; & \frac{\exp(\eta T) - 1}{\eta} L(t)\big[qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)\big] - L(t)dt
\end{aligned}
$$

$$
\text{Therefore } d\Phi(t) \;=\; \Phi(t)\big[qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)L\big] - L(t)dt
$$

$$
\text{where } q \;=\; \frac{\delta(\exp(\eta T) - 1) + \eta}{\exp(\eta T) - 1}
$$

$\square$

4.5 Wealth Valuation of the Investor

Definition 4.5.1. The value of wealth process of the Investor at time t is defined as:

$$V(t) = X^{\Delta,Y,L} + \Psi(t) - \Phi(t) \tag{4.22}$$

The value of wealth $V(T)$ equals the wealth, $X^{\Delta,Y,L}$ plus the discounted expected

stochastic wage income, $\Psi(t)$ less the expeted value of cash outflows $\Phi(t)$.

4.5.1   Proposition 3

The change in wealth of the Investor is given by the dynamics:

$$
\begin{aligned}
dV(t) &= \Big[\Delta^S(t)X^{\Delta,Y(t),L(t)}(\mu - r) + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I\theta^I + X^{\Delta,Y(t),L(t)}r + \Psi(t)F - \Phi(t)\Big]dt \\
&\quad + \Big[\Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_1^S + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I - \Phi(t)\sigma_1^L\Big]dW^I(t) \\
&\quad + \Big[\Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_2^S + \Psi(t)\beta\rho + \Phi(t)\sigma_2^L\Big]dW^S(t) \\
&\quad + \Big[\Psi(t)\beta\sqrt{1 - \rho^2}\Big]dW^w(t) \\
V(0) &= v = x + \Psi(0) - \Phi(0). \tag{4.24}
\end{aligned}
$$

Proof. Taking the differential of both sides of (4.22) and substituting in (4.13), (4.17)

and (4.21), the result follows.

$$
\begin{aligned}
dV(t) \ &= \ dX^{\Delta,Y(t),L(t)} + d\Psi(t) - d\Phi(t) \hspace{4cm} (4.25)\\[2mm]
&= \ \Big[\Delta^S(t)X^{\Delta,Y(t),L(t)}\frac{dS(t)}{S(t)} + \Delta^I(t)X^{\Delta,Y(t),L(t)}\frac{dB(t)}{B(t)}\\[2mm]
&\quad +(1-\Delta^S(t)-\Delta^I(t))X^{\Delta,Y(t),L(t)}\frac{dC(t)}{C(t)} + (Y(t)-L(t))dt\Big]\\[2mm]
&\quad +\Big[\Psi(t)(Fdt + \beta\rho dW^s(t) + \beta\sqrt{1-\rho^2}dW^w(t)) - Y(t)dt\Big]\\[2mm]
&\quad -\Big[\Phi(t)(qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)) - L(t)dt\Big]\\[3mm]
&= \ \Delta^S(t)X^{\Delta,Y(t),L(t)}\big[\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t)\big]\\[2mm]
&\quad +\Delta^I(t)X^{\Delta,Y(t),L(t)}\big[(r+\sigma_I\theta^I)dt + \sigma_I dW^I(t)\big]\\[2mm]
&\quad +X^{\Delta,Y(t),L(t)}(rdt) - \Delta^S(t)X^{\Delta,Y(t),L(t)}(rdt) - \Delta^I(t)X^{\Delta,Y(t),L(t)}(rdt) + Y(t)dt - L(t)dt\\[2mm]
&\quad +\Psi(t)Fdt + \Psi(t)\beta\rho dW^s(t) + \Psi(t)\beta\sqrt{1-\rho^2}dW^w(t) - Y(t)dt\\[2mm]
&\quad +\Phi(t)qdt + \Phi(t)\sigma_1^L dW^I(t) + \Phi(t)\sigma_2^L dW^S(t) + L(t)dt\\[3mm]
&= \ \Delta^S(t)X^{\Delta,Y(t),L(t)}\mu dt + \Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_1^S dW^I(t) + \Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_2^S dW^S(t)\\[2mm]
&\quad +\Delta^I(t)X^{\Delta,Y(t),L(t)}rdt + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I\theta^I dt + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I dW^I(t)\\[2mm]
&\quad +X^{\Delta,Y(t),L(t)}rdt - \Delta^S(t)X^{\Delta,Y(t),L(t)}rdt - \Delta^I(t)X^{\Delta,Y(t),L(t)}rdt + Y(t)dt - L(t)dt\\[2mm]
&\quad +\Psi(t)Fdt + \Psi(t)\beta\rho dW^s(t) + \Psi(t)\beta\sqrt{1-\rho^2}dW^w(t) - Y(t)dt\\[2mm]
&\quad +\Phi(t)qdt + \Phi(t)\sigma_1^L dW^I(t) + \Phi(t)\sigma_2^L dW^S(t) + L(t)dt\\[3mm]
&= \ \Delta^S(t)X^{\Delta,Y(t),L(t)}\mu dt + \Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_1^S dW^I(t) + \Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_2^S dW^S(t)\\[2mm]
&\quad +\Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I\theta^I dt + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I dW^I(t)\\[2mm]
&\quad +X^{\Delta,Y(t),L(t)}rdt - \Delta^S(t)X^{\Delta,Y(t),L(t)}rdt\\[2mm]
&\quad +\Psi(t)Fdt + \Psi(t)\beta\rho dW^s(t) + \Psi(t)\beta\sqrt{1-\rho^2}dW^w(t)
\end{aligned}
$$

$$+\Phi(t)qdt + \Phi(t)\sigma_1^L dW^I(t) + \Phi(t)\sigma_2^L dW^S(t)$$

$$= \left[\Delta^S(t)X^{\Delta,Y(t),L(t)}(\mu - r) + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I\theta^I + X^{\Delta,Y(t),L(t)}r + \Psi(t)Fdt - \Phi(t)q\right]dt$$

$$+\left[\Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_1^S + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I + \Delta^I(t)X^{\Delta,Y(t),L(t)}\sigma_I - \Phi(t)\sigma_1^L\right]dW^I(t)$$

$$+\left[\Delta^S(t)X^{\Delta,Y(t),L(t)}\sigma_2^S + \Psi(t)\beta\rho + \Phi(t)\sigma_2^L\right]dW^S(t)$$

$$+\left[\Psi(t)\beta\sqrt{1 - \rho^2}\right]dW^w(t)$$

$\square$

In this section, we now present the optimal portfolio strategies for the Investor.

4.6   Optimal Portfolio Strategies for the Investor

The Market model has been developed and described in the previous chapters. In this chapter we calculate the optimal portfolio, that is, in this section we consider the optimal portfolio strategy for the Investor.

Dynamic programming is the method that is used to calculate the optimal portfolio choices and the utility function that is used is the power utility. In this case, the power utility function is used as the risk aversion tool. The HJB equation is solved in order to obtain our portfolio strategy.

The main concern in this research is, the Investor is mostly interested in how to allocate the wealth between the risk free asset, the stock and the inflation-linked bond.

4.6.1 Optimal Portfolio

Let $f = C^{1,2}$ and define:

$$G(t) = f(t, V(t)) = f(t, X(t), \Psi(t), \Phi(t)). \tag{4.26}$$

This implies that G(t) is a stochastic process and ;

$$
\begin{aligned}
dG(t) &= \frac{\partial f}{\partial t}(t, X(t), \Psi(t), \Phi(t))dt \\
&+ \frac{\partial f}{\partial X}(t, X(t), \Psi(t), \Phi(t))\big[dX\big] \\
&+ \frac{\partial f}{\partial \Psi}(t, X(t), \Psi(t), \Phi(t))\big[d\Psi\big] \\
&+ \frac{\partial f}{\partial \Phi}(t, X(t), \Psi(t), \Phi(t))\big[d\Phi\big] \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(t, X(t), \Psi(t), \Phi(t))\big[dX\big]^2 \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial \Psi^2}(t, X(t), \Psi(t), \Phi(t))\big[d\Psi\big]^2 \\
&+ \frac{1}{2}\frac{\partial^2 f}{\partial \Phi^2}(t, X(t), \Psi(t), \Phi(t))\big[d\Phi\big]^2
\end{aligned}
$$

$$+\frac{\partial^2 f}{\partial\Psi.\partial X}(t, X(t), \Psi(t), \Phi(t))\big[d\Psi dX\big]$$

$$+\frac{\partial^2 f}{\partial\Phi\partial X}(t, X(t), \Psi(t), \Phi(t))\big[d\Phi dX\big]$$

$$+\frac{\partial^2 f}{\partial\Psi\partial\Phi}(t, X(t), \Psi(t), \Phi(t))\big[d\Psi d\Phi\big] \qquad (4.27)$$

We then take $G = f(t, V(t))$ and substitute on (4.27), we have:

$$
\begin{aligned}
dG(t) \;=\; & \frac{\partial f}{\partial t}(t, X(t), \Psi(t), \Phi(t))dt \\
& +\frac{\partial f}{\partial X}(t, X(t), \Psi(t), \Phi(t))\Big[\Delta^S(t)X^{\Delta,Y,L}(t)[\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t)] \\
& +\Delta^I(t)X^{\Delta,Y,L}(t)[(r + \sigma_I\theta^I)dt + \sigma_I dW^I(t)] \\
& +(1 - \Delta^S(t) - \Delta^I(t))X^{\Delta,Y,L}(t)[rdt] + (Y(t) - L(t))dt\Big] \\
& +\frac{\partial f}{\partial\Psi}(t, X(t), \Psi(t), \Phi(t))\Big[\Psi(t)[Fdt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)] - Y(t)dt\Big] \\
& +\frac{\partial f}{\partial\Phi}(t, X(t), \Psi(t), \Phi(t))\Big[\Phi(t)[qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)] - L(t)dt\Big] \\
& +\frac{1}{2}\frac{\partial^2 f}{\partial X^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Delta^S(t)X^{\Delta,Y,L}(t)[\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t)] \\
& +\Delta^I(t)X^{\Delta,Y,L}(t)[(r + \sigma_I\theta^I)dt + \sigma_I dW^I(t)] \\
& +(1 - \Delta^S(t) - \Delta^I(t))X^{\Delta,Y,L}(t)[rdt] + (Y(t) - L(t))dt\Big]^2 \\
& +\frac{1}{2}\frac{\partial^2 f}{\partial\Psi^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Psi(t)[Fdt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)] - Y(t)dt\Big]^2 \\
& +\frac{1}{2}\frac{\partial^2 f}{\partial\Phi^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Phi(t)[qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)] - L(t)dt\Big]^2 \\
& +\frac{\partial^2 f}{\partial\Psi.\partial X}(t, X(t), \Psi(t), \Phi(t))\Big[\Big(\Psi(t)[Fdt + \beta\rho dW^S(t) + \beta\sqrt{1-\rho^2}dW^w(t)] - Y(t)dt\Big)
\end{aligned}
$$

$$
\cdot \bigg(\Delta^S(t) X^{\Delta,Y,L}(t) \big[\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t) \big] + \Delta^I(t) X^{\Delta,Y,L}(t) \big[(r + \sigma_I \theta^I) dt
$$

$$
+ \sigma_I dW^I(t) \big] + (1 - \Delta^S(t) - \Delta^I(t)) X^{\Delta,Y,L}(t) [rdt] + (Y(t) - L(t)) dt \bigg) \bigg]
$$

$$
+ \frac{\partial^2 f}{\partial \Phi \partial X}(t, X(t), \Psi(t), \Phi(t)) \bigg[\bigg(\Phi(t) [qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)] - L(t) dt \bigg)
$$

$$
\cdot \bigg(\Delta^S(t) X^{\Delta,Y,L}(t) [\mu dt + \sigma_1^S dW^I(t) + \sigma_2^S dW^S(t)] + \Delta^I(t) X^{\Delta,Y,L}(t) [(r + \sigma_I \theta^I) dt
$$

$$
+ \sigma_I dW^I(t)] + (1 - \Delta^S(t) - \Delta^I(t)) X^{\Delta,Y,L}(t) [rdt] + (Y(t) - L(t)) dt \bigg) \bigg]
$$

$$
+ \frac{\partial^2 f}{\partial \Psi \partial \Phi}(t, X(t), \Psi(t), \Phi(t)) \bigg[\bigg(\Psi(t) [Fdt + \beta \rho dW^S(t) + \beta \sqrt{1 - \rho^2} dW^w(t)] - Y(t) dt \bigg)
$$

$$
\cdot \bigg(\Phi(t) [qdt + \sigma_1^L dW^I(t) + \sigma_2^L dW^S(t)] - L(t) dt \bigg) \bigg]
$$

But,

$$
dt.dt = dt.dW^I(t) = dt.dW^S(t) = dW^I(t).dt = 0
$$

$$
dW^I(t).dW^S(t) = dW^S(t).dt = dW^S(t).dW^I(t) = 0
$$

$$
dW^w(t).dW^S(t) = dW^S(t).dW^w = dW^S(t).dW^w(t) = 0
$$

$$
dW^w(t).dW^S(t) = dt.dW^w(t) = dW^w(t).dt = 0
$$

$$
dW^S(t).dW^S = dW^I(t).dW^I(t) = dW^w(t).dW^w(t) = dt
$$

Now,the above simplifies to:

$$
dG(t) \quad = \quad \frac{\partial f}{\partial t}(t, X(t), \Psi(t), \Phi(t)) dt
$$

$$+\frac{\partial f}{\partial X}(t, X(t), \Psi(t), \Phi(t))\Big[\big(\Delta^S x\mu + \Delta^I x\sigma_I\theta^I + xr - \Delta^S xr + Y(t) - L(t)\big)dt$$

$$+\big(\Delta^S x\sigma_1^S + \Delta^I x\sigma_I\big)dW^I(t) + \big(\Delta^S x\sigma_2^S\big)dW^S(t)\Big]$$

$$+\frac{\partial f}{\partial \Psi}(t, X(t), \Psi(t), \Phi(t))\Big[\big(\Psi F - Y(t)\big)dt + \big(\Psi\beta\rho\big)dW^S + \big(\Psi\beta\sqrt{1-\rho^2}\big)dW^w(t)\Big]$$

$$+\frac{\partial f}{\partial \Phi}(t, X(t), \Psi(t), \Phi(t))\Big[\Phi(t)[\big(\Phi q - L(t)\big)dt + \big(\Phi\sigma_1^L\big)dW^I(t) + \big(\Phi\sigma_2^L\big)dW^S(t)\Big]$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial X^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Big((\Delta^S)^2 x^2\big((\sigma_1^S)^2 + (\sigma_2^S)^2\big) + \big(2\Delta^S x\sigma_1^S\Delta^I x\sigma_I\big)$$

$$+\big((\Delta^I)^2 x^2(\sigma_I)^2\big)\Big)dt\Big]$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial \Psi^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Psi^2\beta^2 dt\Big]$$

$$+\frac{1}{2}\frac{\partial^2 f}{\partial \Phi^2}(t, X(t), \Psi(t), \Phi(t))\Big[\Phi^2\big((\sigma_1^L)^2 + (\sigma_2^L)^2\big)dt\Big]$$

$$+\frac{\partial^2 f}{\partial \Psi.\partial X}(t, X(t), \Psi(t), \Phi(t))\Big[\big(\Psi\beta\rho\Delta^S x\sigma_2^S\big)dt\Big]$$

$$+\frac{\partial^2 f}{\partial \Phi\partial X}(t, X(t), \Psi(t), \Phi(t))\Big[\Phi\big(\sigma_1^L\Delta^S x\sigma_1^S + \sigma_1^L\sigma_I + \sigma_2^L\Delta^S x\sigma_2^S\big)dt\Big]$$

$$+\frac{\partial^2 f}{\partial \Psi\partial \Phi}(t, X(t), \Psi(t), \Phi(t))\Big[\big(\Psi\beta\rho\Phi\sigma_2^L\big)dt\Big]$$

Let $f(t, V(t)) = J(t, V(t))$ such that for a given portfolio strategy Δ we introduce the associated utility:

$$J(t, x, \Psi, \Phi, \Delta) = \mathbb{E}_{t,x,\Psi,\Phi}[U(V^\Delta(T))] \tag{4.28}$$

$$dJ(t) \quad = \quad J_t dt$$

$$+ J_x \left[\left(\Delta^S x \mu + \Delta^I x \sigma_I \theta^I + xr - \Delta^S xr + Y(t) - L(t) \right) dt \right.$$

$$\left. + \left(\Delta^S x \sigma_1^S + \Delta^I x \sigma_I \right) dW^I(t) + \left(\Delta^S x \sigma_2^S \right) dW^S(t) \right]$$

$$+ J_\Psi \left[\left(\Psi F - Y(t) \right) dt + \left(\Psi \beta \rho \right) dW^S + \left(\Psi \beta \sqrt{1 - \rho^2} \right) dW^w(t) \right]$$

$$+ J_\Phi \left[\Phi(t) [\left(\Phi q - L(t) \right) dt + \left(\Phi \sigma_1^L \right) dW^I(t) + \left(\Phi \sigma_2^L \right) dW^S(t) \right]$$

$$+ \frac{1}{2} J_{xx} \left[\left((\Delta^S)^2 x^2 \left((\sigma_1^S)^2 + (\sigma_2^S)^2 \right) + \left(2 \Delta^S x \sigma_1^S \Delta^I x \sigma_I \right) + \left((\Delta^I)^2 x^2 (\sigma_I)^2 \right) \right) dt \right]$$

$$+ \frac{1}{2} J_{\Psi\Psi} \left[\Psi^2 \beta^2 dt \right]$$

$$+ \frac{1}{2} J_{\Phi\Phi} \left[\Phi^2 \left((\sigma_1^L)^2 + (\sigma_2^L)^2 \right) dt \right]$$

$$+ J_{\Psi x} \left[\left(\Psi \beta \rho \Delta^S x \sigma_2^S \right) dt \right]$$

$$+ J_{\Phi x} \left[\Phi \left(\sigma_1^L \Delta^S x \sigma_1^S + \sigma_1^L \sigma_I + \sigma_2^L \Delta^S x \sigma_2^S \right) dt \right]$$

$$+ J_{\Psi\Phi} \left[\left(\Psi \beta \rho \Phi \sigma_2^L \right) dt \right]$$

We integrate both sides to get:

$$J(T, V(T)) \;=\; J(t, V(t))$$

$$+ \int_t^T J_s ds$$

$$+ \int_t^T \left[J_x \left(\Delta^S x \mu + \Delta^I x \sigma_I \theta^I + xr - \Delta^S xr + Y(s) - L(s) \right) ds \right.$$

$$\left. + \left(\Delta^S x \sigma_1^S + \Delta^I x \sigma_I \right) dW^I(s) + \left(\Delta^S x \sigma_2^S \right) dW^S(s) \right]$$

$$+ \int_t^T \left[J_\Psi \left(\Psi F - Y(t) \right) dt + \left(\Psi \beta \rho \right) dW^S + \left(\Psi \beta \sqrt{1 - \rho^2} \right) dW^w(s) \right]$$

$$+ \int_t^T \left[J_\Phi \Phi(t)[(\Phi q - L(t))dt + (\Phi \sigma_1^L)dW^I(s) + (\Phi \sigma_2^L)dW^S(s)] \right.$$

$$+ \frac{1}{2} \int_t^T \left[J_{xx} \left((\Delta^S)^2 x^2 ((\sigma_1^S)^2 + (\sigma_2^S)^2) + (2\Delta^S x \sigma_1^S \Delta^I x \sigma_I) + ((\Delta^I)^2 x^2 (\sigma_I)^2) \right) ds \right]$$

$$+ \frac{1}{2} \int_t^T \left[J_{\Psi\Psi} \Psi^2 \beta^2 ds \right]$$

$$+ \frac{1}{2} \int_t^T \left[J_{\Phi\Phi} \Phi^2 ((\sigma_1^L)^2 + (\sigma_2^L)^2) ds \right]$$

$$+ \int_t^T \left[J_{\Psi x} (\Psi \beta \rho \Delta^S x \sigma_2^S) ds \right]$$

$$+ \int_t^T \left[J_{\Phi x} \Phi (\sigma_1^L \Delta^S x \sigma_1^S + \sigma_1^L \sigma_I + \sigma_2^L \Delta^S x \sigma_2^S) ds \right]$$

$$+ \int_t^T \left[J_{\Psi \Phi} (\Psi \beta \rho \Phi \sigma_2^L) ds \right]$$

$$J(T, V(T)) = J(t, V(t))$$

$$+ \int_t^T \left[J_s + (\Delta^S x \mu + \Delta^I x \sigma_I \theta^I + xr - \Delta^S xr + Y(s) - L(s)) J_x \right.$$

$$+ (\Psi F - Y(s)) J_\Psi + (\Phi q - L(s)) J_\Phi$$

$$+ \frac{1}{2} \left((\Delta^S)^2 x^2 ((\sigma_1^S)^2 + (\sigma_2^S)^2) + (2\Delta^S x \sigma_1^S \Delta^I x \sigma_I) + ((\Delta^I)^2 x^2 (\sigma_I)^2) \right) J_{xx}$$

$$+ \frac{1}{2} (\Psi^2 \beta^2) J_{\Psi\Psi} + \frac{1}{2} \Phi^2 ((\sigma_1^L)^2 + (\sigma_2^L)^2) J_{\Phi\Phi} + (\Psi \beta \rho \Delta^S x \sigma_2^S) J_{\Psi x}$$

$$\left. + (\Phi (\sigma_1^L \Delta^S x \sigma_1^S + \sigma_1^L \sigma_I + \sigma_2^L \Delta^S x \sigma_2^S)) J_{\Phi x} + (\Psi \beta \rho \Phi \sigma_2^L) J_{\Psi \Phi} \right] ds$$

$$+ \int_t^T \left[(\Delta^S x \sigma_1^S + \Delta^I x \sigma_I) J_x + (\Phi \sigma_1^L) J_\Phi \right] dW^I(s)$$

$$+ \int_t^T \left[(\Delta^S x \sigma_2^S) J_x + (\Psi \beta \rho) J_\Psi + (\Phi \sigma_2^L) J_\Phi \right] dW^S(s)$$

$$+ \int_t^T \left[(\Psi \beta \sqrt{1 - \rho^2}) J_\Psi \right] dW^w(s) \tag{4.29}$$

Next we take the expectations on both sides of (4.29):

$$
\begin{aligned}
\mathbb{E}_{t,x,\Psi,\Phi}\Big[J(T,V(T))\Big] \;=\;& J(t,V(t)) \\
&+\mathbb{E}_{t,x,\Psi,\Phi}\Bigg[\int_t^T \Big(J_s + \big(\Delta^S x\mu + \Delta^I x\sigma_I\theta^I + xr - \Delta^S xr + Y(s) - L(s)\big)J_x \\
&+\big(\Psi F - Y(s)\big)J_\Psi + \big(\Phi q - L(s)\big)J_\Phi \\
&+\frac{1}{2}\Big((\Delta^S)^2 x^2\big((\sigma_1^S)^2 + (\sigma_2^S)^2\big) + \big(2\Delta^S x\sigma_1^S \Delta^I x\sigma_I\big) + \big((\Delta^I)^2 x^2(\sigma_I)^2\big)\Big) \\
&+\frac{1}{2}\big(\Psi^2\beta^2\big)J_{\Psi\Psi} + \frac{1}{2}\Phi^2\big((\sigma_1^L)^2 + (\sigma_2^L)^2\big)J_{\Phi\Phi} + \big(\Psi\beta\rho\Delta^S x\sigma_2^S\big)J_{\Psi x} \\
&+\big(\Phi\big(\sigma_1^L \Delta^S x\sigma_1^S + \sigma_1^L\sigma_I + \sigma_2^L\Delta^S x\sigma_2^S\big)\big)J_{\Phi x} + \big(\Psi\beta\rho\Phi\sigma_2^L\big)J_{\Psi\Phi}\Big)ds \\
&+\int_t^T \Big(\big(\Delta^S x\sigma_1^S + \Delta^I x\sigma_I\big)J_x + \big(\Phi\sigma_1^L\big)J_\Phi\Big)dW^I(s) \\
&+\int_t^T \Big(\big(\Delta^S x\sigma_2^S\big)J_x + \big(\Psi\beta\rho\big)J_\Psi + \big(\Phi\sigma_2^L\big)J_\Phi\Big)dW^S(s) \\
&+\int_t^T \Big(\big(\Psi\beta\sqrt{1-\rho^2}\big)J_\Psi\Big)dW^w(s)\Bigg]
\end{aligned}
$$

Then:

$$
\begin{aligned}
\mathbb{E}_{t,x,\Psi,\Phi}\Big[J(T,V(T))\Big] \;=\;& J(t,V(t)) \\
&+\mathbb{E}_{t,x,\Psi,\Phi}\Bigg[\int_t^T \Big[J_s + \big(\Delta^S x\mu + \Delta^I x\sigma_I\theta^I + xr - \Delta^S xr + Y(s) - L(s)\big)J_x \\
&+\big(\Psi F - Y(s)\big)J_\Psi + \big(\Phi q - L(s)\big)J_\Phi
\end{aligned}
$$

$$+\frac{1}{2}\left((\Delta^S)^2x^2\big((\sigma_1^S)^2+(\sigma_2^S)^2\big)+\big(2\Delta^Sx\sigma_1^S\Delta^Ix\sigma_I\big)+\big((\Delta^I)^2x^2(\sigma_I)^2\big)\right)J_{xx}$$

$$+\frac{1}{2}\big(\Psi^2\beta^2\big)J_{\Psi\Psi}+\frac{1}{2}\Phi^2\big((\sigma_1^L)^2+(\sigma_2^L)^2\big)J_{\Phi\Phi}+\big(\Psi\beta\rho\Delta^Sx\sigma_2^S\big)J_{\Psi x}$$

$$+\big(\Phi\big(\sigma_1^L\Delta^Sx\sigma_1^S+\sigma_1^L\sigma_I+\sigma_2^L\Delta^Sx\sigma_2^S\big)\big)J_{\Phi x}+\big(\Psi\beta\rho\Phi\sigma_2^L\big)J_{\Psi\Phi}\bigg]ds$$

But because $J(T,V_\Delta(T))=U(V_\Delta(T))$, therefore we have:

$$
\mathbb{E}_{t,x,\Psi,\Phi}\Big[U(T,V(T))\Big] = J(t,x,\Psi,\Phi)
$$

$$+\mathbb{E}_{t,x,\Psi,\Phi}\bigg[\int_t^T\bigg[J_s+\big(\Delta^Sx\mu+\Delta^Ix\sigma_I\theta^I+xr-\Delta^Sxr+Y(s)-L(s)\big)J_x$$

$$+\big(\Psi F-Y(s)\big)J_\Psi+\big(\Phi q-L(s)\big)J_\Phi$$

$$+\frac{1}{2}\left((\Delta^S)^2x^2\big((\sigma_1^S)^2+(\sigma_2^S)^2\big)+\big(2\Delta^Sx\sigma_1^S\Delta^Ix\sigma_I\big)+\big((\Delta^I)^2x^2(\sigma_I)^2\big)\right)J_{xx}$$

$$+\frac{1}{2}\big(\Psi^2\beta^2\big)J_{\Psi\Psi}+\frac{1}{2}\Phi^2\big((\sigma_1^L)^2+(\sigma_2^L)^2\big)J_{\Phi\Phi}+\big(\Psi\beta\rho\Delta^Sx\sigma_2^S\big)J_{\Psi x}$$

$$+\big(\Phi\big(\sigma_1^L\Delta^Sx\sigma_1^S+\sigma_1^L\sigma_I+\sigma_2^L\Delta^Sx\sigma_2^S\big)\big)J_{\Phi x}+\big(\Psi\beta\rho\Phi\sigma_2^L\big)J_{\Psi\Phi}\bigg]ds$$

which implies:

$$
J(t,x,\Psi,\Phi) = \mathbb{E}_{t,x,\Psi,\Phi}\Big[U(T,V(T))\Big]
$$

$$-\mathbb{E}_{t,x,\Psi,\Phi}\left[\int_t^T\left[J_s + \left(\Delta^S x\mu + \Delta^I x\sigma_I\theta^I + xr - \Delta^S xr + Y(s) - L(s)\right)J_x\right.\right.$$

$$+\left(\Psi F - Y(s)\right)J_\Psi + \left(\Phi q - L(s)\right)J_\Phi$$

$$+\frac{1}{2}\left((\Delta^S)^2 x^2\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right) + \left(2\Delta^S x\sigma_1^S\Delta^I x\sigma_I\right) + \left((\Delta^I)^2 x^2(\sigma_I)^2\right)\right)J_{xx}$$

$$+\frac{1}{2}\left(\Psi^2\beta^2\right)J_{\Psi\Psi} + \frac{1}{2}\Phi^2\left((\sigma_1^L)^2 + (\sigma_2^L)^2\right)J_{\Phi\Phi} + \left(\Psi\beta\rho\Delta^S x\sigma_2^S\right)J_{\Psi x}$$

$$+\left(\Phi\left(\sigma_1^L\Delta^S x\sigma_1^S + \sigma_1^L\sigma_I + \sigma_2^L\Delta^S x\sigma_2^S\right)\right)J_{\Phi x} + \left(\Psi\beta\rho\Phi\sigma_2^L\right)J_{\Psi\Phi}\bigg]ds$$

By equation (4.28), we have that:

$$\mathbb{E}_{t,x,\Psi,\Phi}\left[\int_t^T\left[J_s + \left(\Delta^S x\mu + \Delta^I x\sigma_I\theta^I + xr - \Delta^S xr + Y(s) - L(s)\right)J_x\right.\right.$$

$$+\left(\Psi F - Y(s)\right)J_\Psi + \left(\Phi q - L(s)\right)J_\Phi$$

$$+\frac{1}{2}\left((\Delta^S)^2 x^2\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right) + \left(2\Delta^S x\sigma_1^S\Delta^I x\sigma_I\right) + \left((\Delta^I)^2 x^2(\sigma_I)^2\right)\right)J_{xx}$$

$$+\frac{1}{2}\left(\Psi^2\beta^2\right)J_{\Psi\Psi} + \frac{1}{2}\Phi^2\left((\sigma_1^L)^2 + (\sigma_2^L)^2\right)J_{\Phi\Phi} + \left(\Psi\beta\rho\Delta^S x\sigma_2^S\right)J_{\Psi x}$$

$$+\left(\Phi\left(\sigma_1^L\Delta^S x\sigma_1^S + \sigma_1^L\sigma_I + \sigma_2^L\Delta^S x\sigma_2^S\right)\right)J_{\Phi x} + \left(\Psi\beta\rho\Phi\sigma_2^L\right)J_{\Psi\Phi}\bigg]ds = 0$$

The following partial differential equation is obtained by differentiating both sides:

$$\Big[J_t + \big(\Delta^S x \mu + \Delta^I x \sigma_I \theta^I + xr - \Delta^S xr + Y(s) - L(s) \big) J_x$$

$$+ \big(\Psi F - Y(s) \big) J_\Psi + \big(\Phi q - L(s) \big) J_\Phi$$

$$+ \frac{1}{2} \bigg((\Delta^S)^2 x^2 \big((\sigma_1^S)^2 + (\sigma_2^S)^2 \big) + \big(2 \Delta^S x \sigma_1^S \Delta^I x \sigma_I \big) + \big((\Delta^I)^2 x^2 (\sigma_I)^2 \big) \bigg) J_{xx}$$

$$+ \frac{1}{2} \big(\Psi^2 \beta^2 \big) J_{\Psi\Psi} + \frac{1}{2} \Phi^2 \big((\sigma_1^L)^2 + (\sigma_2^L)^2 \big) J_{\Phi\Phi} + \big(\Psi \beta \rho \Delta^S x \sigma_2^S \big) J_{\Psi x}$$

$$+ \big(\Phi \big(\sigma_1^L \Delta^S x \sigma_1^S + \sigma_1^L \sigma_I + \sigma_2^L \Delta^S x \sigma_2^S \big) \big) J_{\Phi x} + \big(\Psi \beta \rho \Phi \sigma_2^L \big) J_{\Psi\Phi} \Big] = 0$$

Consider the value function:

$$V(t, x, \Psi, \Phi) = \sup_\Delta J(t, x, \Psi, \Phi, \Delta)$$

where J is as in equation (4.28).

The value function V satisfies:

$$J_t + \sup_\Delta \Big[\Delta^S x \mu U_x + \Delta^I x \sigma_I \theta^I U_x + xr U_x - \Delta^S xr U_x + Y(t) U_x - L(t) U_x$$

$$+ \Psi F U_\Psi - Y(s) U_\Psi + \Phi q U_\Phi - L(t) U_\Phi$$

$$+ \frac{1}{2} (\Delta^S)^2 x^2 (\sigma_1^S)^2 U_{xx} + + \frac{1}{2} (\Delta^S)^2 x^2 (\sigma_2^S)^2 U_{xx} + \Delta^S x \sigma_1^S \Delta^I x \sigma_I U_{xx} + \frac{1}{2} (\Delta^I)^2 x^2 (\sigma_I)^2 U_{xx}$$

$$+ \frac{1}{2} \Psi^2 \beta^2 U_{\Psi\Psi} + \frac{1}{2} \Phi^2 (\sigma_1^L)^2 U_{\Phi\Phi} + \frac{1}{2} \Phi^2 (\sigma_2^L)^2 U_{\Phi\Phi} + \Psi \beta \rho \Delta^S x \sigma_2^S U_{\Psi x}$$

$$+\Phi\sigma_1^L\Delta^S x\sigma_1^S U_{\Phi x} + \Phi\sigma_1^L\sigma_I U_{\Phi x} + \Phi\sigma_2^L\Delta^S x\sigma_2^S U_{\Phi x} + \Psi\beta\rho\Phi\sigma_2^L U_{\Psi\Phi}\Big] = 0$$

Let our utility function to be:

$$U(T,\nu) = \frac{1}{\gamma}\nu^\gamma$$

We look at the function of Δ which is:

$$
\begin{aligned}
A(\Delta) \quad = \quad & \Delta^S x\mu U_x + \Delta^I x\sigma_I\theta^I U_x - \Delta^S xr u_x \\[2mm]
& +\frac{1}{2}(\Delta^S)^2 x^2(\sigma_1^S)^2 U_{xx} + \frac{1}{2}(\Delta^S)^2 x^2(\sigma_2^S)^2 U_{xx} + \Delta^S x\sigma_1^S\Delta^I x\sigma_I U_{xx} \\[2mm]
& +\frac{1}{2}(\Delta^I)^2 x^2(\sigma_I)^2 U_{xx} + \Psi\beta\rho\Delta^S x\sigma_2^S U_{\Psi x} + \Phi\sigma_1^L\Delta^S x\sigma_1^S U_{\Phi x} \\[2mm]
& +\Phi\sigma_2^L\Delta^S x\sigma_2^S U_{\Phi x} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4.30)
\end{aligned}
$$

We differentiate $U(\nu)$ and substitute in (4.30) to get:

$$
\begin{aligned}
A(\Delta) \quad = \quad & \Delta^S x\mu\big((\nu)^{\nu-1}\big) + \Delta^I x\sigma_I\theta^I\big((\nu)^{\nu-1}\big) - \Delta^S xr\big((\nu)^{\nu-1}\big) \\[2mm]
& +\frac{1}{2}(\Delta^S)^2 x^2(\sigma_1^S)^2\big((\gamma-1).\nu^{\gamma-2}\big) + \frac{1}{2}(\Delta^S)^2 x^2(\sigma_2^S)^2\big((\gamma-1).\nu^{\gamma-2}\big)
\end{aligned}
$$

$$+\Delta^S x \sigma_1^S \Delta^I x \sigma_I \big((\gamma-1).\nu^{\gamma-2}\big) + \frac{1}{2}(\Delta^I)^2 x^2 (\sigma_I)^2 \big((\gamma-1).\nu^{\gamma-2}\big)$$

$$+\Psi\beta\rho\Delta^S x \sigma_2^S \big((\gamma-1).\nu^{\gamma-2}\big) + \Phi\sigma_1^L \Delta^S x \sigma_1^S \big((\gamma-1).\nu^{\gamma-2}\big)$$

$$+\Phi\sigma_2^L \Delta^S x \sigma_2^S \big((\gamma-1).\nu^{\gamma-2}\big) \tag{4.31}$$

Because $A(\Delta)$ is a quassi-concave function of Δ, to obtain its maximum we differentiate (4.31) with respect to Δ^I and Δ^S:

$$\frac{\partial A(\Delta)}{\partial \Delta^I} = x\sigma_I \theta^I (\nu)^{\gamma-1} + \Delta^I x^2 (\sigma_I)^2 (\gamma-1)\nu^{\gamma-2}$$

$$+\Delta^S x \sigma_1^S x \sigma_I (\gamma-1)\nu^{\gamma-2} \tag{4.32}$$

and

$$\frac{\partial A(\Delta)}{\partial \Delta^S} = x\mu(\nu)^{\nu-1} - xr(\nu)^{\gamma-1} + \Delta^S x^2 (\sigma_1^S)^2 (\gamma-1)\nu^{\gamma-2}$$

$$+\Delta^S x^2 (\sigma_2^S)^2 (\gamma-1)\nu^{\gamma-2} + x\sigma_1^S \Delta^I x \sigma_I (\gamma-1)\nu^{\gamma-2}$$

$$+\Psi\beta\rho x \sigma_2^S (\gamma-1)\nu^{\gamma-2} + \Phi\sigma_1^L x \sigma_1^S (\gamma-1)\nu^{\gamma-2}$$

$$+\Phi\sigma_2^L x \sigma_2^S \big((\gamma-1).\nu^{\gamma-2}\big) \tag{4.33}$$

Next, we equate equations (4.32) and (4.33) to zero, and solve for Δ^I and Δ^S respectively.

Solving for Δ^I:

$$x\sigma_I\theta^I(\nu)^{\gamma-1} + \Delta^I x^2(\sigma_I)^2(\gamma-1)\nu^{\gamma-2} + \Delta^S x\sigma_1^S x\sigma_I(\gamma-1)\nu^{\gamma-2} = 0$$

$$\Delta^I x^2(\sigma_I)^2(\gamma-1)\nu^{\gamma-2} = -x\sigma_I\theta^I(\nu)^{\gamma-1} - \Delta^S x\sigma_1^S x\sigma_I(\gamma-1)\nu^{\gamma-2}$$

$$\Delta^I = \frac{-x\sigma_I\theta^I(\nu)^{\gamma-1} - \Delta^S x\sigma_1^S x\sigma_I(\gamma-1)\nu^{\gamma-2}}{x^2(\sigma_I)^2(\gamma-1)\nu^{\gamma-2}}$$

$$\Delta^I = \frac{-x\sigma_I\left(\theta^I(\nu)^{\gamma-1} + \Delta^S\sigma_1^S x(\gamma-1)\nu^{\gamma-2}\right)}{x^2(\sigma_I)^2(\gamma-1)\nu^{\gamma-2}}$$

$$\Delta^I = \frac{-\left(\theta^I(\nu)^{\gamma-1} + \Delta^S\sigma_1^S x(\gamma-1)\nu^{\gamma-2}\right)}{x\sigma_I(\gamma-1)\nu^{\gamma-2}}$$

$$\Delta^I = -\frac{\theta^I(\nu)^{\gamma-1}}{x\sigma_I(\gamma-1)\nu^{\gamma-2}} - \frac{\Delta^S\sigma_1^S x(\gamma-1)\nu^{\gamma-2}}{x\sigma_I(\gamma-1)\nu^{\gamma-2}}$$

$$\Delta^I = -\frac{\theta^I\nu}{x\sigma_I(\gamma-1)} - \frac{\Delta^S\sigma_1^S}{\sigma_I}$$

Solving for Δ^S:

$$x\mu(\nu)^{\nu-1} - xr(\nu)^{\gamma-1} + \Delta^S x^2(\sigma_1^S)^2(\gamma-1)\nu^{\gamma-2} + \Delta^S x^2(\sigma_2^S)^2(\gamma-1)\nu^{\gamma-2}$$

$$+x\sigma_1^S\Delta^I x\sigma_I(\gamma-1)\nu^{\gamma-2} + \Psi\beta\rho x\sigma_2^S(\gamma-1)\nu^{\gamma-2} + \Phi\sigma_1^L x\sigma_1^S(\gamma-1)\nu^{\gamma-2}$$

$$+\Phi\sigma_2^L x\sigma_2^S((\gamma-1).\nu^{\gamma-2} = 0$$

$$\Delta^S x^2(\sigma_1^S)^2(\gamma-1)\nu^{\gamma-2} + \Delta^S x^2(\sigma_2^S)^2(\gamma-1)\nu^{\gamma-2} = -x\mu(\nu)^{\nu-1} + xr(\nu)^{\gamma-1}$$

$$-x\sigma_1^S\Delta^I x\sigma_I(\gamma-1)\nu^{\gamma-2} - \Psi\beta\rho x\sigma_2^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_1^L x\sigma_1^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_2^L x\sigma_2^S((\gamma-1).\nu^{\gamma-2}$$

$$\Delta^S\left(x^2(\sigma_1^S)^2(\gamma-1)\nu^{\gamma-2} + x^2(\sigma_2^S)^2(\gamma-1)\nu^{\gamma-2}\right) = -x\mu(\nu)^{\nu-1} + xr(\nu)^{\gamma-1}$$

$$-x\sigma_1^S\Delta^I x\sigma_I(\gamma-1)\nu^{\gamma-2} - \Psi\beta\rho x\sigma_2^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_1^L x\sigma_1^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_2^L x\sigma_2^S((\gamma-1).\nu^{\gamma-2}$$

$$\Delta^S\left[x^2(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)\right] = -x\mu(\nu)^{\nu-1} + xr(\nu)^{\gamma-1}$$

$$-x\sigma_1^S\Delta^I x\sigma_I(\gamma-1)\nu^{\gamma-2} - \Psi\beta\rho x\sigma_2^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_1^L x\sigma_1^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_2^L x\sigma_2^S((\gamma-1).\nu^{\gamma-2}$$

$$\Delta^S = \frac{-x\mu(\nu)^{\nu-1} + xr(\nu)^{\gamma-1} - x\sigma_1^S\Delta^I x\sigma_I(\gamma-1)\nu^{\gamma-2}}{x^2(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\Psi\beta\rho x\sigma_2^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_1^L x\sigma_1^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_2^L x\sigma_2^S((\gamma-1).\nu^{\gamma-2}}{x^2(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$\Delta^S = \frac{-\mu(\nu)^{\nu-1}}{x(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{r(\nu)^{\gamma-1}}{x(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\sigma_1^S\Delta^I\sigma_I(\gamma-1)\nu^{\gamma-2}}{(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\Psi\beta\rho\sigma_2^S(\gamma-1)\nu^{\gamma-2}}{x(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\Phi\sigma_1^L\sigma_1^S(\gamma-1)\nu^{\gamma-2} - \Phi\sigma_2^L\sigma_2^S(\gamma-1)\nu^{\gamma-2}}{x(\gamma-1)\nu^{\gamma-2}\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$\Delta^S = \frac{-\mu(\nu)}{x(\gamma-1)\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{r(\nu)}{x(\gamma-1)\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\sigma_1^S\Delta^I\sigma_I}{\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\Psi\beta\rho\sigma_2^S}{x\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S}{x\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$\Delta^S = \frac{-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S}{x\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\sigma_1^S\Delta^I\sigma_I}{\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)} + \frac{(-\mu+r)\nu}{x(\gamma-1)\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)} \qquad (4.34)$$

Therefore,

$$\Delta^I = -\frac{\theta^I\nu}{x\sigma_I(\gamma-1)} - \frac{\Delta^S\sigma_1^S}{\sigma_I} \qquad (4.35)$$

Take (4.35) into (4.34):

$$\Delta^S = \frac{-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S}{x\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\sigma_1^S\left[-\frac{\theta^I\nu}{x\sigma_I(\gamma-1)} - \frac{\Delta^S\sigma_1^S}{\sigma_I}\right]\sigma_I}{\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)} + \frac{(-\mu+r)\nu}{x(\gamma-1)\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$\Delta^S = \frac{-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S}{x\left((\sigma_1^S)^2 + (\sigma_2^S)^2\right)}$$

$$+\frac{-\sigma_1^S\left[-\frac{\theta^I\nu}{x(\gamma-1)}-\Delta^S\sigma_1^S\right]}{\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)}+\frac{(-\mu+r)\nu}{x(\gamma-1)\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)}$$

$$\Delta^S=\frac{\left[-\Psi\beta\rho\sigma_2^S-\Phi\sigma_1^L\sigma_1^S-\Phi\sigma_2^L\sigma_2^S\right](\gamma-1)+\left(-\sigma_1^S\left[-\frac{\theta^I\nu}{x(\gamma-1)}-\Delta^S\sigma_1^S\right]x(\gamma-1)\right)+(-(\mu+r)\nu)}{x(\gamma-1)\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)}$$

$$\Delta^S=\frac{\left[-\Psi\beta\rho\sigma_2^S-\Phi\sigma_1^L\sigma_1^S-\Phi\sigma_2^L\sigma_2^S\right](\gamma-1)+\sigma_1^S\theta^I\nu+\Delta^S(\sigma_1^S)^2x(\gamma-1)+(-(\mu+r)\nu)}{x(\gamma-1)\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)}$$

$$\Delta^S\left[x(\gamma-1)\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)\right]=\left[-\Psi\beta\rho\sigma_2^S-\Phi\sigma_1^L\sigma_1^S-\Phi\sigma_2^L\sigma_2^S\right](\gamma-1)+\sigma_1^S\theta^I\nu$$

$$+\Delta^S(\sigma_1^S)^2x(\gamma-1)+(-(\mu+r)\nu)$$

$$\Delta^S\left[x(\gamma-1)\left((\sigma_1^S)^2+(\sigma_2^S)^2\right)\right]-\Delta^S(\sigma_1^S)^2x(\gamma-1)=\left[-\Psi\beta\rho\sigma_2^S-\Phi\sigma_1^L\sigma_1^S-\Phi\sigma_2^L\sigma_2^S\right](\gamma-1)$$

$$+\sigma_1^S\theta^I\nu+(-(\mu+r)\nu)$$

$$\Delta^S\left[x(\gamma-1)\right]\left((\sigma_1^S)^2+(\sigma_2^S)^2-(\sigma_1^S)^2\right)=\left[-\Psi\beta\rho\sigma_2^S-\Phi\sigma_1^L\sigma_1^S-\Phi\sigma_2^L\sigma_2^S\right](\gamma-1)+\sigma_1^S\theta^I\nu+(-(\mu+r)\nu)$$

$$\Delta^S = \frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I\nu + (-(\mu+r)\nu)}{\left[x(\gamma-1)\right]\left((\sigma_2^S)^2\right)} \tag{4.36}$$

Take (4.36) into (4.35):

$$\Delta^I = -\frac{\theta^I\nu}{x\sigma_I(\gamma-1)} - \frac{\left[\frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I\nu + (-(\mu+r)\nu)}{\left[x(\gamma-1)\right]\left((\sigma_2^S)^2\right)}\right]\sigma_1^S}{\sigma_I}$$

and

$$\Delta^S = \frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I\nu + (-(\mu+r)\nu)}{\left[x(\gamma-1)\right]\left((\sigma_2^S)^2\right)}$$

$$\begin{pmatrix}\Delta^I \\ \\ \Delta^S\end{pmatrix} = \begin{pmatrix}-\dfrac{\theta^I\nu}{x\sigma_I(\gamma-1)} - \dfrac{\left[\frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I\nu(t) + (-(\mu+r)\nu(t))}{\left[x(\gamma-1)\right]\left((\sigma_2^S)^2 - (\sigma_1^S)^2\right)}\right]\sigma_1^S}{\sigma_I} \\ \\ \dfrac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I\nu(t) + (-(\mu+r)\nu(t))}{\left[x(\gamma-1)\right]\left((\sigma_2^S)^2\right)}\end{pmatrix}$$

$$(\Delta^I)^* = -\frac{\theta^I V^*(t)}{X^*\sigma_I(\gamma-1)} - \frac{\left[\frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I V^*(t) + (-(\mu+r)V^*(t))}{\left[X^*(\gamma-1)\right]\left((\sigma_2^S)^2\right)}\right]\sigma_1^S}{\sigma_I} \tag{4.37}$$

and

$$(\Delta^S)^* = \frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I V^*(t) + (-(\mu+r)V^*(t))}{\left[X^*(\gamma-1)\right]\left((\sigma_2^S)^2\right)}$$

(4.38)

But,

$$(\Delta_0)^* = 1 - (\Delta^I)^* - (\Delta^S)^*$$

This then implies that:

$$
\begin{aligned}
(\Delta^0)^* &= 1 - \left[-\frac{\theta^I V^*(t)}{X^*\sigma_I(\gamma-1)} - \frac{\left[\frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I V^*(t) + (-(\mu+r)V^*(t))}{\left[X^*(\gamma-1)\right]\left((\sigma_2^S)^2\right)}\right]\sigma_1^S}{\sigma_I}\right] \\
&\quad - \left[\frac{\left[-\Psi\beta\rho\sigma_2^S - \Phi\sigma_1^L\sigma_1^S - \Phi\sigma_2^L\sigma_2^S\right](\gamma-1) + \sigma_1^S\theta^I V^*(t) + (-(\mu+r)V^*(t))}{\left[X^*(\gamma-1)\right]\left((\sigma_2^S)^2\right)}\right]
\end{aligned}
$$

(4.39)

Chapter 5

Discussion and Results

5.1   Discussion

In this thesis, we studied the Optimal investment under inflation protection and optimal portfolio with stochastic wage income and cash outflows. First we derived the dynamics of the stochastic wage income and stochastic cash outflows. Then we derived the dynamics of the wealth process,the dynamics of the expected discounted stochastic wage income process at time t and the expected discounted cash outflows process at time t. Then, the value of the wealth process of the Investor at time t was found. Lastly, we calculated the optimal portfolio strategies for the Investor.

5.1.1   General assumptions

When carrying out this research, some assumptions were needed, and these are:

1. For the stochastic income over time t, $\alpha(Y_t, t) = \alpha Y_t$ and $\beta(Y_t, t) = \beta Y_t$, and the stochastic wage income follows a geometric Brownian motion with constant volatility β.

2. The discounted cash outflows follow the geometric Brownian motion defined by a constant volatility $\sigma^L = (\sigma_1^L, \sigma_2^L)$.

3.We assumed there were no jumps. That is, we considered the stock price to be that of a geometric Brownian motion, but in actual fact the jumps exist as the price changes.

5.2   Results

The value of expected future stochastic wage income process $\Psi(t)$ is proportional to the instantaneous total stochastic wage income process $Y(t)$. This can be seen by equation (4.16).

Another observation is that the expected discounted cash process $\Phi(t)$ is proportional to the instantaneous total cash outflows process $L(t)$ and this is seen by (4.20).

The change in wealth of the investor was also obtained and this is seen by equation

(4.23).

In other words, some of the main results are Proposition 1, Proposition 2 and Proposition 3

The other main results are (4.37), (4.38) and (4.39), that is, the optimal portfolio of the Investor in the inflation linked bond, in the stock market and in the cash account respectively, were found.

This work can be extended by using a numerical approach where our formulas which are being developed can be applied. Also, further developments can be used by using a different utility function. More general results can be achieved if we assume that the stock price process are driven by semi-martingales or processes with jumps.

Chapter 6

Conclusion

In this dissertation, the optimal portfolios with stochastic wag income and stochastic cash outflows for an Investor were studied. The optimal share of portfolios in stock and inflation-linked bond depend on stochastic wage income, stochastic cash outflows and the optimal wealth level of the investment at time t. Also, as the market evolves, parts of the portfolio values in stock and inflation-linked bond should be transferred to the cash account. This work can be extended by using a different utility function and assuming that the stock price process are driven by semi-martingales or processes with jumps.

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