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On the Subspace Problem for Quasi-normed Spaces

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

A definitive positive answer to the so-called *proper subspace problem* (a.k.a. the *atomic space problem*) for quasi-normed spaces is given: every infinite dimensional quasi-normed space has a proper closed infinite-dimensional subspace.

Keywords: Quasi-Banach spaces; Hahn-Banach extension property; Minkowski functional; semi-norm.

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1 Introduction and Main Result

Although the first research paper on quasi-normed spaces appeared in the early 1940's, the systematic study of these spaces only really started in the late 1950's and early 1960's with the work of Klee, Peck, Rolewitz, and Zezalko. The paper of Duren, Romberg, and Shields [1] opened up many new directions, leading to a significant increase of activity in the 1970's and 1980's. The results arguably brought about new and important contributions to our appreciation of the Banach space theory. The most recent comprehensive study of the quasi-normed spaces was given by Kalton [2] in early 2000's. Unfortunately, the interest had been since somewhat subsided, leaving one of the remaining long standing structural problems for quasi-normed spaces unresolved; namely, the *proper subspace problem*:

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Problem 1.1. *Does every quasi-normed space admit a proper infinite-dimensional subspace?*

Such a problem was first raised in 1960's by Pelczynski (cf. [3]). It has since relayed on by several authors in the intervening years (see e.g. [4, 2, 5, 6, 7] and references therein). For a recent contribution in the more general context of F-spaces (complete linear metric spaces) see [6]. It is arguably the most fundamental structure problem for quasi-normed spaces, or more generally for F-spaces. Despite the fact that many progresses have been carried out in understanding the structure of subspaces of quasi-normed spaces, the answer to the above Problem 1.1 remains elusive. Unlike the case of normed space where the Hahn-Banach Theorem ensures the existence of a rich enough class of continuous linear functionals, which implies the existence of a very rich class of infinite dimensional closed subspaces, not enough satisfactory results are known for the corresponding problem for quasi-normed spaces.

The best known classical examples of quasi-normed spaces are H_p, ℓ_p, and $L_(p)$ $0 < p < 1$. It turns out that all of these spaces happen to have proper infinite dimensional subspaces. Such a property is less of a surprise for H_p, and $\ell_p, 0 < p < 1$ as both types of these spaces have separating duals. On the other hand, for $0 < p < 1$, it is also known that although L_p has the highly undesirable property of having only trivial dual, that is $L_p^* = 0$, L_p does admit proper infinite dimensional subspaces. Kalton noticed in [2] that such a result, which is due to Day [8], can be extended to any space that fails to have a separating dual. Kalton (see e.g.[2]) gave an equivalent formulation of the above problem known as the atomic problem. A quasi-normed space is said to be atomic if it admits no proper closed infinite dimensional subspaces. Problem 1.1 can be restated as follows:

Problem 1.2. *Do atomic quasi-normed spaces exist?*

It is clear that if a space has a basic sequence then it has a proper infinite dimensional subspace; namely, the closed linear span of the basic sequence. Therefore, an atomic space (if it exists) obviously cannot contain any basic sequence. Kalton [9] proved that if a F-space has strictly weaker Hausdorff vector topology, then it contains basic sequences. On the other hand, Kalton [10] also showed how to construct a quasi-Banach space without a basic sequence. Kalton's results did not give a definitive answer to the atomic problem. It has not yet been clear whether or not the example in [9] can be used to construct an atomic space. More recently, Barroso [11] obtained an improvement to Kalton's result by showing that a quasi-Banach space contains a basic sequence if and only if it contains a countably infinite dimensional subspace whose dual is separating. A more recent study by Botelho et al [12] brought about some partial answers to the problem. Related results can also be found in [13]. For result related to the existence of basic sequences, the reader is referred to e.g. [14].

In this short note, we attempt to prove the following theorem which gives a definitive positive answer to the subspace problem: atomic spaces do not exist.

Theorem 1.3. *Every infinite dimensional quasi-Banach space admits proper closed infinite dimensional subspaces.*

2 Proof of the Main Result

For simplicity, we shall only consider real vector spaces. The results carry through to the complex vector spaces with standard slight alteration.

Let us recall that a vector space X is a quasi-normed space if it is equipped with a functional $q : X \to [0, \infty]$ satisfying

1. $q(x) > 0$ if $x \neq 0$,

2. $q(\alpha x) = |\alpha| q(x)$ for every scalar α and for every $x \in X$,
3. There exists a constant $k \geq 1$ such that $q(x+y) \leq k(q(x) + q(y))$ for all $x, y \in X$.

The smallest constant k satisfying condition (3) is referred to as the *modulus of concavity* of the quasi-norm of U. Sets of the form

$$B(y,r) = \{x \in X : q(x-y) < r\}$$

shall be termed as q-balls centered at y with radius r. It is known that if a quasi-normed space (X,q) has a separating dual, then one can associate a norm on X by the formula

$$||x||_c = \sup\{|f(x)| : f \in X^*\}$$

where X^* denote the space of all bounded linear functionals on X. This is the largest norm on X dominated by q. Hence, $\|\cdot\|_c$ induces a weaker Hausdorff norm-topology on X. The completion X_c of X with this norm is known as the *Banach envelop* of X (see e.g. [2]).

On the other hand, a quasi-normed space X that admits trivial dual (i.e. $X^* = 0$) cannot be normable: if it were, the Hahn-Banach Theorem (HBT) would hold, contradicting $X^* = 0$. Nonetheless, as we shall see below, it is always possible to naturally associate to any quasi-normed space a locally convex Hausdorff topology that is weaker than the original q-topology.

Recall that a *seminorm* on a vector space X is a functional $p : X \to [0, \infty]$ satisfying:

1. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
2. $q(\alpha x) = |\alpha| q(x)$ for every scalar α and for every $x \in X$.

A family P of seminorms on X is said to be *separating* if for every $x \neq 0$ in X, there exists $p \in P$ such that $p(x) \neq 0$. It is a well-known fact that every locally convex topological vector space X possesses a family of separating semi-norms. Namely, if $\mathcal{B}$ is a local base of convex, balanced and absorbing neighborhoods of X, then the family of Minkowski functionals $\{p_B : B \in \mathcal{B}\}$ forms a separating family of continuous seminorms on X Recall that the Minkowski functional associated to an absorbing convex set B is defined on X by $p_B(x) = \inf\{t > 0 : x \in tB\}$.

In what follows, given a subset B of X, we shall denote by $\widehat{B}$ the convex hull of B, i.e. the set of all convex combinations of elements of B.

Proposition 2.1. *Let (X,q) be a quasi-normed space with modulus of concavity k. Let $p_{\widehat{B}}$ denote the Minkowski functional associated to the convex hull $\widehat{B}$ of the balanced and absorbing set $B = \{x \in X : q(x) < r\}$, where $r \in (0, \infty)$, that is $p_{\widehat{B}}(x) = \inf\{s > 0 : s^{-1}x \in \widehat{B}\}$. Then $p_{\widehat{B}} \leq q \leq kp_{\widehat{B}}$.*

Proof. Clearly, if $q(x) < s$ then $p_{\widehat{B}}(x) < s$, i.e. $p_{\widehat{B}} \leq q$. On the other hand, if $p_{\widehat{B}}(x) < \infty$ then there exists $0 < s < p_{\widehat{B}}(x)$ such that $s^{-1}x \in \widehat{B}$. That is, $x = s(ty + (1-t)z)$ where $y, z \in B$ and $t \in [0,1]$. It follows that

$$q(x) \leq sq(ty + (1-t)z) \leq sk(tq(y) + (1-t)q(z)) \leq kp_{\widehat{B}}(x),$$

i.e. $q \leq kp_{\widehat{B}}$. The proof is complete. □

Theorem 2.1. *Let (X,q) be a quasi-normed space with modulus of concavity k. Let $\mathcal{B}$ be the set of all q-balls of X. For every $B \in \mathcal{B}$, let $\widehat{B}$ the convex hull of B. Then $\widehat{B} = \{x \in X : p_{\widehat{B}}(x) < 1\}$ and $p_{\widehat{B}} B \in \mathcal{B}$ is a family of continuous separating seminorms.*

Proof. Continuity follows from the above proposition. Let us show that $\{p_{\widehat{B}} : B \in B\}$ is separating. If $x \neq 0$, then there exists $n > 0$ such that $q(x) > 1/n$. Since $B\widehat{(0,1/n)}$ is convex, the set of $0 \leq \lambda$ such that $\lambda x \in B\widehat{(0,1/n)}$ is an interval containing 0. Since it does not contain 1, it must be bounded. Hence, $p_{\widehat{B}}(x) = \inf\{t > 0 : x \in tB\widehat{(0,1/n)}\}$ is strictly positive. □

Completeness is crucial for our next result.

Theorem 2.2. *Let (X, q) be a quasi-Banach space. Let $\{p_\alpha : \alpha \in A\}$ be a family of continuous semi-norms on X. Then the set*

$$E = \left\{ x \in X : \sup_{\alpha \in A} p_\alpha(x) < \infty \right\}$$

is of second category if and only if $E = X$.

Proof. We only need to show the necessity. Fix $\epsilon > 0$. Consider the closed set

$$F = \bigcap_{\alpha \in A} \{x \in X : p_\alpha(x) \leq \varepsilon/2\}.$$

If $x \in E$, one can choose n large enough so that $\sup_{\alpha \in A} \{p(x)\} < n/2$. It follows that $E \subset \bigcup_{n \in \mathbb{N}} nF$. Since E is of second category, at least one of the nF is of second category. It follows that F has an interior point, say x. Thus there exists a q-neighborhood V of zero such that $x + V \subset F$. Then for every $\alpha \in A$, and for every $v \in V$ we have

$$p_\alpha(v) = p_\alpha(x + v - x) \leq p_\alpha(x + v) + p_\alpha(x) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Now, let $x \in X$. Choose $r > 0$ so that $r^{-1}x \in V$. Then for every $\alpha \in A$ we have

$$r^{-1} p_\alpha(x) = p_\alpha(r^{-1}x),$$

from which it follows that $x \in E$. Hence, $E = X$. The proof is complete. □

It follows that if (X, q) is a quasi-Banach space, then the one can associate a family of continuous seminorms given by the Minkowski's functional of the convex hull of the q -balls of X. Such a family then generates a topology on X that is weaker than the original q-topology of X. More formally, we have

Theorem 2.3. *Let (X, q) be a quasi-Banach space. Then we can associate a weaker locally convex topology on X defined by the norm*

$$\|x\|_q = \sup_{B \in \mathcal{B}} p_{\widehat{B}}(x),$$

where $\mathcal{B}$ is the set of all the q-balls of X, and $\widehat{B}$ is the convex hull of B, for every $B \in \mathcal{B}$.

It is no longer difficult to indicate a proof for our main result Theorem 1.3.

Proof. Assume to the contrary that X is atomic. Then X must be minimal in the sense that it cannot have a strictly weaker Hausdorff vector topology. Thus the $\|\cdot\|_q$-topology must coincide with the original quasi-norm topology. Hence (X, q) is locally convex, that is, (X, q) is a Banach space. Contradiction! The proof is complete. □

3 Remarks on Hahn-Banach Theorem

The classical Hahn-Banach Theorem assures us that the dual space of a non-trivial normed linear space is itself non-trivial.

Theorem 3.1. *(Hahn-Banach) If f is a bounded linear functional on a subspace of a normed linear space, then f extends to the whole space with preservation of norm.*

Note that virtually no hypotheses are required beyond linearity and the existence of a norm. Extension properties can also be stated for dominated linear functionals on a subspace of a quasi-normed space but without preservation of the quasi-norm. For a quasi-normed space (X, q), and a subspace U of X, let us agree to say that a linear functional $f : U \to \mathbb{R}$ is q-dominated, if there exists $c > 0$ such that $\|f(x)\| \leq cq(x)$ for all xU. The set of all q-dominated functionals on U shall be denoted by $U^{(q)}$ and the norm of $f \in U^{(q)}$ is defined to be

$$\|f\|_{U^{(q)}} = \inf \{c > 0 : |f(x)| \leq cq(x), x \in U\}.$$

A straightforward adaptation of the proof of the elementary classical Hahn-Banach Theorem yields the following.

Theorem 3.2. *Let (X, q) be a quasi-normed space with modulus of concavity k. Let U be a subspace of X, and let $f \in X^{(q)}$. Assume that $x_0 \in X \backslash U$. Then there exists a linear functional $\bar{f} : U + \mathbb{R}x_0 \to \mathbb{R}$ such that $\bar{f}(u) = f(u)$ for all $u \in U$ and $\|\bar{f}\|_{(U+\mathbb{R}x_0)^{(q)}} \leq k \|f\|_{U^{(q)}}$.*

Proof. We observe that for $u, u' \in U$,

$$f(u) - f(u') \leq f(u - u') \leq q(u - u') \leq k(q(x_0 + u) + q(x_0 + u')).$$

Such an inequality can be written as

$$-f(u') - kq(u' + x_0) \leq f(u) + kq(u + x_0).$$

We can choose a number θ such that

$$\sup \{-f(u') - kq(u' + x_0) : u \in U\} \leq \theta \leq \inf \{f(u) + kq(u + x_0) : u \in U\}.$$

We then define $\bar{f} : U + \mathbb{R}x_0 \to \mathbb{R}$ by $\bar{f}(u + tx_0) = f(u) + t\theta$. Clearly, $\bar{f}(u) = f(u)$ for all $u \in U$. We claim that $\bar{f}(u + tx_0) \leq kq(u + tx_0)$ for all $u \in U$ and for all $t \in \mathbb{R}$. Note that the inequality is true for $t = 0$. For $t \neq 0$, we have

$$\bar{f}(u + tx_0)| = |f(u) + t\theta| = |t||f(u) + \theta| \leq k|t|q(u/t + x_0) = kq(u + tx_0)$$

Hence, our claim. The proof is complete. □

Obviously, the above extension can be iterated on any finite dimensional extension of the subspace U to obtain the following:

Theorem 3.3. *Let (X, q) be a quasi-normed space with modulus of concavity k. Let U be a subspace of X, and F a subspace of X such that $\dim F = n$ and $U \cap F = 0$. Then every $f \in U^{(q)}$ admits a linear extension $\bar{f} : U \oplus F \to \mathbb{R}$ satisfying $\|\bar{f}\|_{(U\oplus F)^{(q)}} \leq k^n \|f\|_{U^{(q)}}$.*

On the other hand, due to the lack of norm-preservation, nothing guarantees that an infinite iteration of the extension method in Theorem 3.2 will yield a q-dominated linear functional on the whole space. To obtain a q-dominated extension on the whole space, an infinite iteration of the above method would require that the sequence $n \mapsto k^n$ be bounded. That can only happen if the modulus of concavity k of the quasi-norm is equal to 1, that is to say, if X is in fact already a normed space.

Duren et al [1] defined a quasi-normed space (or more generally F-space) X to have the Hahn-Banach Extension Property (HBEP) if whenever f is a linear functional quasi-bounded on a closed subspace of X, then f has a q-dominated linear extension on X. After noticing that all the classical examples of F-spaces lack the HBEP, they asked whether or not it is true that an F-space has the HBEP if and only if it is locally convex. Kalton [10] showed that a quasi-Banach space has HBEP if and only if it is a Banach space.

On the other hand, it is now clear from our result that the existence of proper closed infinite dimensional subspaces is not related to the HBT. The above results and discussion lead us to formulate the following definition:

Definition 3.1. Let (X, q) be a quasi-normed space. A closed linear subspace U of X is said to be a *HB-subspace* if every $f \in U^{(q)}$ can be extended to an element of $X^{(q)}$.

It is clear that a quasi-normed space (X, q) has the HBEP, if and only if every closed subspace of X is an HB-subspace. Hence, if (X, q) is a quasi-Banach space which is not locally convex, then it must admit a non-HB-subspace. Day's result [8] shows that it is possible to find a non-HB-subspace in L^p, $0 < p < 1$. As already noticed by Kalton (see for example [2]) , Day's construction will work on any space which fails to have a separating dual. It is clear that a quasi-normed space with trivial dual cannot have any HB-subspace. Hence, any proper closed infinite dimensional subspace of a quasi-normed space with trivial dual is a non-HB-subspace.

On the other hand, even in quasi-normed spaces with separating dual such as the sequence spaces ℓ^p, $0 < p < 1$, it is possible to find non-HB-subspaces (see for example [7]).

Theorem 3.4. *If a quasi-normed space (X, q) admits an infinite dimensional HB-subspace, then it admits an infinite dimensional subspace with separating family of functionals in $X^{(q)}$.*

Proof. Assume that U is an infinite dimensional HB-subspace of X. Let $u_1 \in U$, $u_1 \neq 0$, and let $f_1 \in X^{(q)}$ such that $f_1(u_1) \neq 0$. Then $\ker f_1$ cannot have a trivial dual. Fix $u_2 \in \ker f_1$ and let $f_2 \in X^{(q)}$ such that $f_2(u_2) \neq 0$. Continuing in this way, we obtain a sequence of vectors $\{u_n : n \in \mathbb{N}\}$ in X and a sequence of functionals $\{f_n : n \in \mathbb{N}\} \in X^{(q)}$. It is clear that the later sequence is a separating family for the span of the former sequence. □

Finally, Duren et al [1] also defined the notion of *a proper closed weakly dense (PCWD) subspace* as a proper closed subset F of X such that the quotient X/F has trivial dual. It is easily seen that if X has PCWD, then it must have a non-HB subspace.

4 Conclusions

This note is entirely devoted to the study of infinite dimensional subspaces of (non-locally convex) topological vector spaces. We gave a definitive answer to the long standing problem of existence of infinite dimensional subspaces for quasi-Banach spaces. Altough our proofs was done for the special case of quasi-Banach spaces, they can easily be extended to the more general case of F-spaces (complete linear metric spaces).

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Competing Interests

Author has declared that no competing interests exist.

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