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Extensions of the Lusin's Theorem, the Severini-Egorov's Theorem and the Riesz Subsequence Theorems

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We give extensions of the Lusin's Theorem, the Severini-Egorov's Theorem, and the Riesz Subsequence Theorems to the setting of a non-additive vector valued set functions and sequences of functions taking values in general metric spaces.

Keywords: Convergence theorems; Lusin's theorem; Severini-Egorov's theorem; Borel Cantelli's Lemma; Riesz subsequence theorems, vector valued set functions.

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1 Introduction

The Lusin's Theorem (LT) and the Severini-Egorov's Theorem (SET) are respectively the second and the third of Littlewood's three principles of real analysis. Informally, the LT states that every measurable function is nearly continuous and the SET gives a condition for the uniform convergence of a pointwise convergent sequence of measurable functions. These two theorems gave

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rise to number of important developments in mathematical analysis and its application have been subjects of various extensions.

The SET was first proved by the Italian mathematician Carlo Severini in 1910 [1], and again one year later independently by the Russian mathematician Dimitri Egorov [2]. The LT was first proved by Nikolai Lusin [3] in 1912 as an application of the SET. In 1916, Lusin [4] succeeded in slightly relaxing the requirement of finiteness of measure of the domain of convergence of the pointwise converging functions in the statement of the SET. The first mathematicians to prove independently the SET in the nowadays common abstract measure space setting were Frigyes Riesz (1922, 1928) [5], and in Wacaw Sierpiski (1928) [6].

In this paper, we attempt to prove that results related to the LT and SET theorems can either be sustained or naturally extended to the more general setting of functions taking values in topological spaces or metric spaces. Furthermore, such generalizations are done under significantly relaxed requirements on the set functions; namely, the set functions are no longer required to be real valued; we also replace additivity assumption with subadditivity. The proofs of most of our results are adapted from the proofs of the classical LT and SVT that can be seen in most Real Analysis or Measure and Integration Theory textbooks, see e.g. [7], [8].

This paper will be organized as follows. In Section 2, we define and discuss the notion of integrator space which generalizes the concept of measurable space. We carry out natural extension of the LT and the SET respectively in Section 3 and Section 4. Further extension of the SET is discussed in Section 5. Finally, an extended version of the Riesz Subsequence Theorem will be given in Section 6.

2 Integrator Space

Throughout this note, Ω denotes our basic set; the set of all subsets of will be denoted by 2^{Ω}. We shall always consider a system $\mathcal{E}$ of subsets of 2^{Ω} that contains the empty set $\emptyset$.

By an **integrato**r we mean a set function $\boldsymbol{\mu} : \mathcal{E} \to Y$, taking values in a given real or complex normed vector space $(Y, \|\cdot\|_Y)$ and satisfying:

1. $\boldsymbol{\mu}(\emptyset) = 0$;
2. $\|\boldsymbol{\mu}(A)\|_Y \leq \|\boldsymbol{\mu}(B)\|_Y$ whenever $A \subset B$ in $\mathcal{E}$ (monotone);
3. $\left\|\boldsymbol{\mu}(\bigcup_{i\in I} A_i)\right\|_Y \leq \sum_{i\in I} \|\boldsymbol{\mu}(A_i)\|_Y$ for every finite set I (subadditive).

We additively extend the set function $\boldsymbol{\mu}$ as follows

$$\boldsymbol{\mu}(A \bigcup B) = \boldsymbol{\mu}(A) + \boldsymbol{\mu}(B)$$

whenever A, B are disjoint subsets of $\mathcal{E}$. It is easily checked that the obtained extension is an integrator on the ring $r(\mathcal{E})$ of subsets generated by the set system $\mathcal{E}$.

For example, the additive extension of the length function on bounded intervals defines an integrator on the ring generated by bounded intervals of the real number system.

We say that the set system in $\mathcal{E}$ is a σ-**covering system** if for every subset A of Ω, there exists a sequence $\{E_n\}$ of elements of $\mathcal{E}$ such that $A \subset \bigcup_{n\in\mathbb{N}} E_n$. For example, the semiring of intervals is a σ-covering system for the set $\mathbb{R}$ of real numbers. In such a case, we define the $\boldsymbol{\mu}$-size of a subset A of Ω to be

$$\boldsymbol{\mu}^*(A) = \inf\left\{\sum_{n\in\mathbb{N}} \|\boldsymbol{\mu}(E_n)\|_Y : A \subset \bigcup_{n\in\mathbb{N}} E_n\right\}$$

where the infimum is taken over all sequences $\{E_n\} \subset \mathcal{E}$ such that $A \subset \bigcup_{n\in\mathbb{N}} E_n$. In what follows we always consider a set system $\mathcal{E}$ that is σ-covering on the basic set Ω. We notice that the function $A \mapsto \boldsymbol{\mu}^*(A)$ itself is a non-negative integrator on the power set 2^Ω.

An example of σ-covering system is given by the collection of all bounded intervals, (resp. rectangles, (resp. boxes)) on the real line (resp. plane, (resp. space)).

A triplet $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ consisting of a nonempty set Ω, a σ-covering set system $\mathcal{E}$, and an integrator $\boldsymbol{\mu}$ will be called an **integrator space**.

For example, the triplet $(\mathbb{R}, r(\mathcal{I}), \ell)$ where $r(\mathcal{I})$ is the ring generated by the bounded intervals and ℓ is the additive extension of the interval length function, is an integrator space. Likewise, the triplet $(\mathbb{R}, \mathcal{E}, \boldsymbol{\lambda}^*)$ where $\mathcal{E}$ is any σ-algebra contained in the power set of the real number system, $\boldsymbol{\lambda}^*$ is the Lebesgue outer measure, is an example of integrator space. The Lebesgue measure space is the integrator space $(\mathbb{R}, \mathcal{E}, \boldsymbol{\lambda})$ where $\mathcal{E}$ the σ-algebra of the Lebesgue measurable sets and $\boldsymbol{\lambda}$ is the classical Lebesgue measure. As we have mentioned above, if $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ is an integrator space, then so is $(\Omega, 2^\Omega, \boldsymbol{\mu}^*)$.

A full integration theory has been developed using such a notion, and examples of applications can be seen in [9],[10],[11].

3 Extension of Lusin's Theorem

We say that an integrator space $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ is **topological** if (Ω, τ_Ω) is a topological space and the set system $\mathcal{E}$ contains the topology τ_Ω of Ω. Recall that a function f is continuous from a topological space (Ω, τ_Ω) into another topological space (Γ, τ_Γ) if for every open set $O \in \tau_\Gamma$, the inverse image $f^{-1}(O) \in \tau_\Omega$. In this section, we fix a topological integrator space $(\Omega, \mathcal{E}, \boldsymbol{\mu})$. Given another topological space (Γ, τ_Γ), we say that a function $f : (\Omega, \tau_\Omega) \to (\Gamma, \tau_\Gamma)$ is

- **$\boldsymbol{\mu}$-nearly continuous** over Ω if for every open set $O \in \tau_\Gamma$, given arbitrary $\varepsilon > 0$, there exists $U_\varepsilon \in \tau_\Omega$ such that $U_\varepsilon \supset f^{-1}(O)$ and $\boldsymbol{\mu}^*(U_\varepsilon \setminus f^{-1}(O)) < \varepsilon$.
- **$\boldsymbol{\mu}$-almost continuous** if for every $\varepsilon > 0$, there is a set $B \subset A$ with $\boldsymbol{\mu}^*(B) < \varepsilon$, so that the restriction of f to $A \setminus B$ is continuous.

For example, if λ denotes the Lebesgue measure on $\mathbb{R}$, then a λ-measurable real valued function defined on a measurable subset of $\mathbb{R}$ is λ-nearly continuous. The classical LT states that a Lebesgue real valued function defined on a measurable set of finite measure is λ-almost continuous.

The following theorem is a version of the LT in the setting of a vector valued integrator.

Theorem 3.1. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be a topological integrator space. Then a $\boldsymbol{\mu}$-nearly continuous function $f : (\Omega, \tau_\Omega) \to (\Gamma, \tau_\Gamma)$ is $\boldsymbol{\mu}$-almost continuous.*

Proof. Fix $\varepsilon > 0$. Let $\{O_n : n{\in}N\}$ be a countable base for the topology τ_Γ. For each $n \in \mathbb{N}$, take $U_n \in \tau_\Omega$ such that $U_n \supset f^{-1}(O_n)$ and $\boldsymbol{\mu}^*(U_n \setminus f^{-1}(O_n)) < 2^{-n}\varepsilon$. For $n = 1$, we let

$$A_1 = \Omega \setminus (U_1 \setminus f^{-1}(O_1)).$$

Proof. Then we have $f^{-1}(O_1) = A_1 \bigcap U_1$ which is clearly an open set on the relative topology on A_1. For $n = 1$, we let

$$A_2 = A_1 \setminus (U_2 \setminus f^{-1}(O_2)).$$

Then we have $f^{-1}(O_2) = A_2 \bigcap U_2$ which is clearly an open set on the relative topology on A_2. We repeat the process successively for all $n \in N$. We then set

$$A = \Omega \setminus (\bigcup_{n\in\mathbb{N}} U_n \setminus f^{-1}(O_n)).$$

Clearly, the restriction of f to A is continuous and

$$\boldsymbol{\mu}^*(\Omega \setminus A) \leq \sum_{n\in\mathbb{N}} \boldsymbol{\mu}^*(U_n \setminus f^{-1}(O_n)) < \sum_{n\in\mathbb{N}} 2^{-n}\varepsilon < \varepsilon.$$

The proof is complete. □

4 Extension of Severini-Egorov's Theorem

The classical SET states that almost everywhere convergent sequences of measurable functions on a finite measure space converge almost uniformly; that is, for every $\varepsilon > 0$ the convergence is uniform on a set whose complement has measure less than ε, cf. [5].

We consider an integrator space $(\Omega, \mathcal{E}, \boldsymbol{\mu})$. Recall that a sequence $n \mapsto f_n$ of functions defined on a subset $A \in 2^\Omega$ and taking value in a metric space (X, d) is said

- to **converge $\boldsymbol{\mu}$-almost everywhere** to a function $f : A \to X$ if there exists a set $B \subset A$, of $\boldsymbol{\mu}$-size zero such that the sequence $n \mapsto f_n(\omega)$ for every $\omega \in A \setminus B$.
- to **converge uniformly** on A to a function $f : A \to X$ if for every $\varepsilon > 0$, there exists $N > 0$ such that $\sup \{d(f(\omega), f_n(\omega)) : \omega \in A\} < \varepsilon$ whenever $n > N$.
- to be **uniformly Cauchy** on A if for every $\varepsilon > 0$, there exists $N > 0$ such that whenever $n > N$ and for every $p \in \mathbb{N}$, one has $\sup \{d(f_{n+p}(\omega), f_n(\omega)) : \omega \in A\} < \varepsilon$.
- to **converge $\boldsymbol{\mu}$-almost uniformly** on A to a function $f : A \to X$ if for every $\varepsilon > 0$, there exists $B \subset A$ with $\boldsymbol{\mu}^*(B) < \varepsilon$ such that the sequence $n \to f_n$ converges uniformly on the set $A \setminus B$.
- to be **$\boldsymbol{\mu}$-almost uniformly Cauchy** on A if for every $\varepsilon > 0$, there exists $B \subset A$ with$\boldsymbol{\mu}^*(B) < \varepsilon$ such that the sequence $n \mapsto f_n$ is uniformly Cauchy on the set $A \setminus B$.

We now state and prove the following natural extension of the SET.

Theorem 4.1. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. If a subset A of Ω is of finite $\boldsymbol{\mu}$–size, and if $n \to f_n$ is a sequence of X-valued functions converging $\boldsymbol{\mu}$-almost everywhere on A to an X-valued function f then the sequence $n \to f_n$ converges almost uniformly to f on A.*

That is, loosely speaking, if one has $\boldsymbol{\mu}$-almost everywhere convergence, one can get uniform convergence by cutting out a part of arbitrarily small size of the domain.

Proof. First, we may assume the convergence of the sequence $n \to f_n$ is everywhere by cutting out a set of $\boldsymbol{\mu}$-size zero. Next we define the sets

$$A_n^m := \bigcap_{i=n}^{m} \left\{ \omega \in \Omega : d(f(\omega), f_n(\omega)) < \frac{1}{m} \right\}.$$

As n gets bigger, we are taking the intersection of fewer and fewer sets, and so $A_1^m \subset A_2^m \subset \cdots$. Since the sequence $n \to f_n$ converges pointwise to f, eventually $d(f(\omega), f_n(\omega)) < \frac{1}{m}$. □

5 Further Extensions

We introduce some notions that will be useful in obtaining further generalization of the SET. Again we fix an integrator space $(\Omega, \mathcal{E}, \boldsymbol{\mu})$, a metric space (X, d), a sequence $n \mapsto f_n$ of X-valued functions defined on Ω, and f an X-valued functions also defined on Ω. For $n \in N$,$r > 0$, and A a subset of Ω, we let

$$R(f_n, f, r, A) = \bigcup_{k=n} \omega \in A : d(f(\omega), f_n(\omega)) > r.$$

Then it is quickly seen that

- for fixed $r > 0$, the sequence $n \mapsto R(f_n, f, r, A)$ is non-increasing.
- for fixed $n \in \mathbb{N}$, the mapping $r \mapsto R(f_n, f, r, A)$ is non-increasing on $[0, \infty)$.

Our next two results give characterizations of uniform convergence and almost uniform convergence.

Theorem 5.1. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. Let $n \mapsto f_n$ be a sequence of X-valued functions on a subset A of Ω. Then the sequence $n \mapsto f_n$ converges uniformly on A to a function $f : A \to X$ if and only if for every A, there exists $n \in \mathbb{N}$ such that $R(f_n, f, r, A) = \emptyset$.*

Proof. If the sequence $n \mapsto f_n$ converges uniformly to f, then given $r > 0$ there exists $n_r \in \mathbb{N}$ such that

$$\sup d(f(\omega), f_n(\omega)) : \omega \in A \leq r$$

whenever $n \geq n_r$. Therefore $R(f_n, f, r, A) = \emptyset$. Conversely, assume that given $r > 0$ there exists $n_r \in \mathbb{N}$ such that $R(f_n, f, r, A) = \emptyset$. Then for $n \geq n_r$ we have $\sup d(f(\omega), f_n(\omega)) : \omega \in A \leq r$, that is, the sequence $n \mapsto f_n$ converges uniformly to f. □

Theorem 5.2. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. Let $n \mapsto f_n$ be a sequence of X-valued functions on a subset A of Ω. Then the sequence $n \mapsto f_n$ converges $\boldsymbol{\mu}$-almost uniformly on A to a function $f : A \to X$ if and only if for every $r > 0$, the sequence $n \mapsto \boldsymbol{\mu}\left(R(f_n, f, r, A)\right)$ converges to 0.*

Proof. Assume that the sequence $n \mapsto f_n$ converges $\boldsymbol{\mu}$-almost uniformly on A to a function $f : A \to X$. Fix $r > 0$. Given $\varepsilon > 0$, there exists $B \subset A$ with $\boldsymbol{\mu}^*(B) < \varepsilon$ such that the sequence $n \mapsto f_n$ converges uniformly on the set $A \setminus B$. Therefore, there exists $n_r \in \mathbb{N}$ such that $R(f_n, f, r, A) \subset B$. It follows that whenever $n \geq n_r$, we have

$$\boldsymbol{\mu}^*\left(R(f_n, f, r, A)\right) \leq \boldsymbol{\mu}^*\left(R(f_{n_r}, f, r, A)\right) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that the sequence $n \mapsto \boldsymbol{\mu}^*\left(R(f_n, f, r, A)\right)$ converges to 0. Conversely, assume that given $r > 0$, the sequence $n \mapsto \boldsymbol{\mu}^*\left(R(f_n, f, r, A)\right)$ converges to 0. Let $\varepsilon > 0$. For each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$\boldsymbol{\mu}^*\left(R(f_{n_k}, f, \frac{1}{k}, A)\right) < \frac{\varepsilon}{2^k}.$$

Let $B = \bigcup_{k \in \mathbb{N}} R(f_{n_k}, f, \frac{1}{k}, A)$. Then

$$\boldsymbol{\mu}^*(B) \leq \sum_{k=1}^{\infty} \boldsymbol{\mu}^*\left(R(f_{n_k}, f, \frac{1}{k}, A)\right) \leq \varepsilon.$$

Since $R(f_{n_k}, f, \frac{1}{k}, A) \subset B$, whenever $n \geq n_k$ and $\omega \notin B$, we have $d(f(\omega), f_n(\omega)) < 1/k$. Thus the sequence $n \mapsto f_n$ converges uniformly on the set $A \setminus B$. □

For our next results, we need the following easy extension of the Borel-Cantelli Lemma.

Theorem 5.3. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and $n \mapsto E_n$ a sequence in 2^Ω. If the series $\sum_{n=1}^{\infty} \mu^*(E_n)$ converges then the sequence $n \mapsto \boldsymbol{\mu}^*\left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_n\right)\right)$ converges to 0.*

Proof. Assume that the series $\sum_{n=1}^{\infty} \boldsymbol{\mu}^*(E_n)$ converges. By rearranging, we can assume that the sequence $n \mapsto \boldsymbol{\mu}^*(E_n)$ is in non-increasing order. Since the series $\sum_{n=1}^{\infty} \boldsymbol{\mu}^*(E_n)$ converges, the sequence $n \mapsto \sum_{k=n}^{\infty} \boldsymbol{\mu}^*(E_k)$ converges to 0. It follows that

$$\boldsymbol{\mu}^* \left(\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} E_k \right) \right) \leq \inf_{n \geq 1} \boldsymbol{\mu}^* \left(\bigcup_{k=n}^{\infty} E_k \right) \leq \inf_{n \geq 1} \sum_{k=n}^{\infty} \boldsymbol{\mu}^* \left(E_k \right) = 0.$$

The proof is complete. □

We are now ready to state and prove a generalized version of the SET.

Theorem 5.4. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. Let $n \mapsto f_n$ be a sequence of X-valued functions on a subset A of Ω. Then the following statements are equivalent:*

1. *The sequence $n \mapsto f_n$ converges $\boldsymbol{\mu}$-almost uniformly on A to a function $f : A \to X$.*
2. *For every $r > 0$, $n \mapsto \boldsymbol{\mu}^* (R(f_n, f, r, A))$ converges to 0.*
3. *For every $r > 0$, there exists $n \in \mathbb{N}$ such that $\boldsymbol{\mu}^* (R(f_n, f, r, A)) < \infty$ and that the sequence $n \mapsto f_n$ converges $\boldsymbol{\mu}$-almost everywhere on A.*

Proof. Only the equivalence 2. ⇔ 3. remains to be shown. Since convergent sequences are bounded, the property in 2. implies that for every $r > 0$, we have $\boldsymbol{\mu}^* (R(f_n, f, r, A)) < \infty$, and thus the first part of the statement in 3. is satisfied.

For each $n \in \mathbb{N}$, let $E_n \subset A$ such that $\mu^*(E_n) < 1/n$ and that the sequence $n \mapsto f_n$ converges uniformly to f on the set $A \setminus E_n$. Let $F := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. Then by Theorem 5.3, $\mu^*(F) = 0$. If $\omega \in A \setminus E_n$, then ω belongs to only finitely many E_n, and thus $\omega \in A \setminus E_n$ for all n greater than some integer $n(\omega)$. Hence $f_n(\omega) \to f(\omega)$ for $\omega \in A \setminus F$.

Conversely, let F be a subset of A of $\boldsymbol{\mu}$-size 0 such that $f_n(\omega) \nrightarrow f(\omega)$ for $\omega \in F$. Fix $r > 0$. We claim that $E := \bigcap_{n=1}^{\infty} R(f_{n_r}, f, r, A) \subset F$. Indeed, if $\omega \in E$, then for every $n \in \mathbb{N}$ there exists $k \geq n$ such that $d(f(\omega), f_k(\omega)) > r$. Thus $\omega \in F$ as claimed. Hence

$$0 \leq \lim_{n \to \infty} \boldsymbol{\mu}^*(R(f_n, f, r, A)) \leq \boldsymbol{\mu}^*(E) \leq \boldsymbol{\mu}^*(F) = 0,$$

showing that 2. is satisfied. □

6 Extension of the Riesz Subsequence Theorems

In ths secion, we discuss further relations between different modes of convergence of sequences of functions, extending the classical theroems due to Riesz. Again, the triplet $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ is an integrator space, and the pair (X, d) is a metric space. We say that a sequence of X-valued functions $n \mapsto f_n$ defined on Ω

- **$\boldsymbol{\mu}$-converges** to an X-valued function f defined on $A \subset \Omega$ provided that for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mu^*(\omega \in A : d(f_n(\omega), f(\omega)) \geq \varepsilon) = 0.$$

  We then write $f_n \xrightarrow{\boldsymbol{\mu}} f$.

- **$\boldsymbol{\mu}$-Cauchy** on $A \subset \Omega$ provided that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and for every $p \in \mathbb{N}$, one has

$$\lim_{n \to \infty} \mu^*(\omega \in A : d(f_{n+p}(\omega), f_n(\omega)) \geq \varepsilon) = 0.$$

It immediately follows from the subadditivity of the set function $\boldsymbol{\mu}$ that if the sequence $n \mapsto f_n$ is $\boldsymbol{\mu}$-convergent then it is $\boldsymbol{\mu}$-Cauchy.

Theorem 6.1. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. Let $n \mapsto f_n$ be a sequence of X-valued functions on a subset A of Ω converging $\boldsymbol{\mu}$-almost uniformly to a function f on A. Then $f_n \overset{\boldsymbol{\mu}}{\to} f$.*

Proof. Let $r > 0$. For every $p \in \mathbb{N}$ there exists $E_p \subset A$ with $\boldsymbol{\mu}^*(E_p) < 1/p$ such that the sequence $n \mapsto f_n$ converges uniformly to f on the set $A \setminus E_p$. It follows that there exists $n(r,p) \in \mathbb{N}$ such that whenever $n \geq n(r,p)$, we have

$$\mu^*(\omega \in A : d(f_n(\omega), f(\omega)) \geq \varepsilon) \leq \mu^*(E_p) < \frac{1}{p}.$$

Since p is arbitrary, we infer that $f_n \overset{\boldsymbol{\mu}}{\to} f$. □

Theorem 6.2. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a complete metric space. Let $n \mapsto f_n$ be a $\boldsymbol{\mu}$-Cauchy sequence of X-valued functions on a subset A of Ω. Then there exists a subsequence $k \mapsto f_{n_k}$ that is*

1. *$\boldsymbol{\mu}$-almost everywhere convergent,*
2. *$\boldsymbol{\mu}$-almost uniformly convergent, and*
3. *$\boldsymbol{\mu}$-convergent to a function $f : A \to X$.*

Proof. Assume that the sequence $n \mapsto f_n$ is $\boldsymbol{\mu}$-Cauchy. Then for every $r > 0$, there exists $n(r) \in \mathbb{N}$ such that whenever $n \geq n(r)$ and $p \in \mathbb{N}$, we have

$$\boldsymbol{\mu}^*(\omega \in A : d(f_(n+p)(\omega), f_n(\omega)) \geq r) \leq r.$$

We let $n_1 = n(\frac{1}{2})$. Inductively, we define $n_{k+1} := max n_k + 1, n(\frac{1}{2^k})$. Then the subsequence $k \mapsto g_k := f_{n_k}$ satisfies

$$\boldsymbol{\mu}^*(\omega \in A : d(g_(k+1)(\omega), g_k(\omega)) \geq \frac{1}{2^k}) \leq \frac{1}{2^k}.$$

Let $F := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$. By Theorem , $\boldsymbol{\mu}^*(F) = 0$. For $\omega \in A \setminus F$, there exists $k_\omega \in \mathbb{N}$ such that $\omega \in A \setminus E_k$ for all $k \geq k_\omega$. It follows that whenever $m \geq k_\omega$ and for every $p \in \mathbb{N}$ we have

$$\begin{aligned} d(g_(m+p)(\omega), g_m(\omega)) &\leq d(g_{m+1}(\omega), g_m(\omega)) + \cdots + d(g_{m+p}(\omega), g_{m+p-1}(\omega)) \\ &\leq \frac{1}{2^k} + \cdots + \frac{1}{2^{m+p-1}} < \frac{1}{2^{m-1}}. \end{aligned}$$

Since (X, d) is complete, it follows that the subsequence $k \mapsto g_k(\omega)$ converges for $\omega \in A \setminus F$. If we now define

$$f(\omega) = \begin{cases} \lim_{k\to\infty} g_k(\omega) & \text{if } \omega \in A \setminus F \\ 0 & \text{otherwise,} \end{cases}$$

then $k \mapsto g_k(\omega)$ converges $\boldsymbol{\mu}$-almost everywhere to f. We next show that the sequence $k \mapsto g_k$ converges $\boldsymbol{\mu}$-almost uniformly to f. Let $r > 0$ and $N \in \mathbb{N}$ large enough so that $2^{-N+1} < r$. We then let $F_N := \bigcup_{k=N}^{\infty} E_k$. Then $\boldsymbol{\mu}^*(F_N) \leq 2^{-N+1} < r$. If $\omega \in A \setminus F_N$ then $\omega \in A \setminus E_k$ for all $k \geq N$. It follows that whenever $i \geq N$ and for every $p \in \mathbb{N}$, we have

$$d(g_{i+p}(\omega), g_i(\omega)) \leq 2^{-i+1} < 2^{-N+1}.$$

Passing to a limit as $p \to \infty$, we have for $i \geq N$

$$d(f(\omega), g_i(\omega)) \leq 2^{-i+1} < 2^{-N+1} < r.$$

Since $r > 0$ is arbitrary, the sequence $k \mapsto g_k$ converges $\boldsymbol{\mu}$-almost uniformly to f.

Finally, an application of Theorem 6.1 shows that $f_{n_k} \xrightarrow{\boldsymbol{\mu}} f$. The proof is complete. □

More can be said under the hypothesis of Theorem 6.2. Noticing that

$$\{\omega \in A : d(f(\omega), f_n(\omega)) > r\} \subset \{\omega \in A : d(f(\omega), f_{n_k}(\omega)) > r\} \cup \{\omega \in A : d(f_{n_k}(\omega), f_n(\omega)) > r\},$$

we have

$$\begin{aligned} \boldsymbol{\mu}^* (\{\omega \in A : d(f(\omega), f_n(\omega)) > r\}) &\leq \boldsymbol{\mu}^* (\{\omega \in A : d(f(\omega), f_{n_k}(\omega)) > r\}) \\ &\quad + \boldsymbol{\mu}^* \{\omega \in A : d(f_{n_k}(\omega), f_n(\omega)) > r\}. \end{aligned}$$

Since the sequence $f_{n_k} \xrightarrow{\boldsymbol{\mu}} f$ and the sequence $n \mapsto f_n$ be a $\boldsymbol{\mu}$-Cauchy, the two terms on the right-hand side converges to 0. Hence, we can state the following theorem.

Theorem 6.3. *Let $(\Omega, \mathcal{E}, \boldsymbol{\mu})$ be an integrator space and (X, d) a metric space. Every $\boldsymbol{\mu}$-Cauchy sequence of X-valued functions on a subset A of Ω is $\boldsymbol{\mu}$-convergent on A.*

7 Conclusions

This work presents a comprehensive view of the natural extensions of the Lusin's Theorem, Severini-Egorov's Theorem, and the Riesz Subsequence Theorem to the setting of functions taking values in metric spaces or topological spaces. The strength of such extensions is the fact that the functions are no longer assumed to be measurable and the set functions need not be additive nor non-negative real-valued. The present note contributes to the strengthening of the unifying property of the new approach to integration theory introduced and disseminated in [9],[10],[11].

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Severini C. Sulle successioni di funzioni ortogonali. On sequences of orthogonal functions, Atti dell'Accademia Gioenia.; serie 5a, Memoria XIII. 1910;3(5):17.

[2] Egoroff, D. Th. Sur les suites des fonctions mesurables, Comptes rendus hebdomadaires des sances de l'Acadmie des sciences. 1911;152:244-246.

[3] Lusin N. Sur les proprits des fonctions mesurables, Comptes Rendus Acad. Sci. Paris. 1912;154:1688-1690.

[4] Lusin N. Integral and trigonometric series. Matematicheskii Sbornik (in Russian). 1916; 30(1):1242

[5] Riesz F. Elementarer Beweis des Egoroffschen Satzes. Monatshefte fur Mathematik und Physik (in German). 1928;35(1):243-248.

[6] Sierpiski W. Remarque sur le theoreme de M. Egoroff, Comptes Rendus des sances de la Socit des Sciences et des Lettres de Varsovie. 1928;21:84-87.

[7] Bartle RG. A general bilinear vector integral. Studia Math. 1956;15:337-352. Zbl 0070.28102.

[8] Rao MM. Measure Theory and integration. 2nd edition, Marcel Dekker Inc. New-York, Basel; 2004.

[9] Robdera MA. Unified approach to vector valued integration, International Journal of Functional Analysis. Operator Theory and Application. 2013;5(2):119-139.

[10] Robdera MA. Tensor integral: A comprehensive approach to the integration theory. British Journal of Mathematics and Computer Science. 2014;4(22):3236-3244.

[11] Robdera MA. On non-metric covering lemma and extended Lebesgue differentiation theorem. British Journal of Mathematics and Computer Science. 2015;8(3):220-228.

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