Exponential polynomials on commutative hypergroups

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Abstract. Polynomials and exponential polynomials play a fundamental role in the theory of spectral analysis and spectral synthesis on commutative groups. Recently several new results have been published in this field ([2], [3], [4], [6]). Spectral analysis and spectral synthesis has been studied on some types of commutative hypergroups, as well. However, a satisfactory definition of exponential monomials on general commutative hypergroups has not been available so far. In [5], [7], [8] and [9] the authors use a special concept on polynomial and Sturm–Liouville-hypergroups. Here we give a general definition, which covers the known special cases.

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In this paper $\mathbb{C}$ denotes the set of complex numbers. By a *hypergroup* we mean a locally compact hypergroup. If K is a commutative hypergroup, then $\mathcal{C}(K)$ denotes the locally convex topological vector space of all continuous complex valued functions defined on K, equipped with the pointwise operations and the topology of uniform convergence on compact sets. The involution on K induces an involution on $\mathcal{C}(K)$ in the following manner: if the involution on K is denoted by $\check{}$, then we define $\check{f}(x) = f(\check{x})$ for each f in $\mathcal{C}(K)$ and x in K.

The dual of $\mathcal{C}(K)$ can be identified with $\mathcal{M}_c(K)$, the space of all compactly supported complex measures on K, and the pairing between $\mathcal{C}(K)$ and

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$\mathcal{M}_c(K)$ is given by the formula

$$\langle \mu, f\rangle = \int f\, d\mu\,.$$

Convolution on $\mathcal{M}_c(K)$ is, as usually, defined by

$$\mu * \nu(x) = \int \mu(x * y)\, d\nu(y)$$

for any μ, ν in $\mathcal{M}_c(K)$ and x in K. Convolution converts the linear space $\mathcal{M}_c(K)$ into a commutative algebra with unit δ_e, e being the identity in K.

We also define convolution of measures in $\mathcal{M}_c(K)$ with arbitrary functions in $\mathcal{C}(K)$ by the same formula

$$\mu * f(x) = \int f(x * \check{y})\, d\mu(y)$$

for each μ in $\mathcal{M}_c(K)$, f in $\mathcal{C}(K)$ and x in K. The linear operators $f \mapsto \mu * f$ on $\mathcal{C}(K)$ are called *convolution operators.*

Translation with the element y in K is the operator mapping the function f in $\mathcal{C}(K)$ onto its *translate* $\tau_y f$ defined by $\tau_y f(x) = f(x * y)$ for any x in K. Clearly, τ_y is a convolution operator, namely, it is the convolution with the measure $\delta_{\check{y}}$. A subset of $\mathcal{C}(K)$ is called *translation invariant*, if it contains all translates of its elements. A closed linear subspace of $\mathcal{C}(K)$ is called a *variety* on K, if it is translation invariant. For each function f the smallest variety containing f is called the *variety generated by* f and is denoted by $\tau(f)$. It is the intersection of all varieties containing f.

For basic knowledge on hypergroups the reader is referred to [1], [8].

A basic function class is formed by the joint eigenfunctions of all translation operators, that is, by those nonzero continuous functions $\varphi : K \to \mathbb{C}$ satisfying

$$\tau_y \varphi = m(y) \cdot \varphi \tag{1}$$

with some $m : K \to \mathbb{C}$, that is

$$\varphi(x * y) = m(y)\varphi(x) \tag{2}$$

for all x, y in K. It follows

$$\varphi(y) = \varphi(e) \cdot m(y)\,,$$

which implies $\varphi(e) \neq 0$, consequently, by (2), we have

$$m(x * y) = m(x)m(y) \tag{3}$$

for all x, y in K. Nonzero continuous functions $m : K \to \mathbb{C}$ satisfying (3) for each x, y in K are called *exponentials.* Clearly, every exponential generates a one dimensional variety, and conversely, every one dimensional variety is generated by an exponential.

Using translations one introduces *difference operators* $\Delta_y = \tau_y - \tau_e$ and higher order difference operators $\Delta_{y_1,y_2,\dots,y_n} = \Pi_{i=1}^n \Delta_{y_i}$ for each $y_1, y_2 \dots, y_n$ in K. Obviously, $\Delta_{y_1,y_2,\dots,y_n}$ is a convolution operator, namely

$$\Delta_{y_1,y_2,\dots,y_n} f = \Pi_{i=1}^n (\delta_{\check{y}_i} - \delta_e) * f\,,$$

where Π is meant as a convolution product.

Difference operators, in particular, higher order difference operators are related to another important function classes on commutative topological groups. We intend to consider these classes on commutative hypergroups as well. A continuous function $f : K \to \mathbb{C}$ is called a *generalized polynomial*, if there is a natural number n such that

$$\Delta_{y_1,y_2,\dots,y_{n+1}} f = 0 \tag{4}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in K. In this case we say that f is *of degree at most* n and the *degree* of f is the smallest n for which f is of degree at most n.

A continuous homomorphism of K in the additive group of complex numbers is called an *additive function*. Clearly, every nonzero additive function is a generalized polynomial of degree 1.

A generalized polynomial is called simply a *polynomial* if it generates a finite dimensional variety. It is known that if G is a commutative topological group, then a complex valued function on G is a polynomial if and only if it is a polynomial of additive functions. In other words, polynomials on commutative groups are exactly the elements of the complex function algebra generated by the constants and the additive functions and they are characterized in the class of generalized polynomials by the property generating a finite dimensional variety. Hence it seems to be reasonable to use this property for their definition on commutative hypergroups.

We shall also use *modified difference operators* defined as follows: given an exponential m, a continuous function f and an element y in K, then we let

$$\Delta_{m;y} f(x) = f(x+y) - m(y)\, f(x)$$

for each x in K. The iterates are defined for any positive integer n and for each $y_1, y_2, \dots, y_n$ in K by

$$\Delta_{m;y_1,y_2,\dots,y_n} = \Pi_{i=1}^n \Delta_{m;y_i}\,.$$

Obviously, these operators are also convolution operators, namely

$$\Delta_{m;y_1,y_2,\dots,y_n} f = \Pi_{i=1}^n (\delta_{\check{y}_i} - m(y_i)\, \delta_e) * f$$

holds. On the right hand side Π is meant as a convolution. In particular, for $m = 1$ we have $\Delta_{1;y_1,y_2,\dots,y_n} = \Delta_{y_1,y_2,\dots,y_n}$.

Modified difference operators are related to another basic function class on commutative topological groups. Namely, a continuous function is called an *exponential monomial*, if it is the product of a polynomial and an exponential. Linear combinations of exponential monomials are called *exponential* *polynomials*. Further, a continuous complex valued function f is called a *generalized exponential monomial*, if it is the product of a generalized polynomial and an exponential, and linear combinations of generalized monomials are called *generalized exponential polynomials*. Unfortunately, these concepts cannot be used in this form in the hypergroup-situation. The reason is that pointwise multiplication of function values is in general not compatible with the linear character of the convolution defined on hypergroups. In particular, the pointwise product of two exponentials is not necessarily an exponential and the product of two additive functions is in general not biadditive. Nevertheless, for general purposes it would be desirable to introduce a reasonable concept of exponential monomial on commutative hypergroups. We made some attempts in doing this on particular hypergroups, like polynomial and Sturm–Liouville hypergroups (see [5], [7], [8]). The point of this paper is to offer a general definition, which seems to work on arbitrary locally compact commutative hypergroups and which reduces to the usual concept on commutative topological groups.

Let K be a commutative hypergroup. Then the continuous function $\varphi : K \to \mathbb{C}$ is called *generalized exponential monomial*, if there exists an exponential m on K and a natural number n such that

$$\Delta_{m;y_1,y_2,\dots,y_{n+1}}\varphi(x) = 0 \tag{5}$$

holds for each $y_1, y_2, \dots, y_{n+1}$ in K. The smallest n with this property is called the *degree* of φ. If (5) holds, then we say that the generalized exponential monomial φ *corresponds to the exponential* m. We note that we do not claim or require that m is unique. A generalized exponential monomial is called simply an *exponential monomial*, if it generates a finite dimensional variety. A linear combination of generalized exponential monomials, or exponential monomials is called *generalized exponential polynomial*, or *exponential polynomial*, respectively.

By our above remark, every generalized polynomial is a generalized exponential monomial, and every polynomial is an exponential monomial corresponding to the exponential identically 1. Every exponential is an exponential monomial of degree 0 and every additive function is an exponential polynomial of degree 1. The following theorem shows that this concept of exponential monomials and generalized exponential monomials is compatible with the corresponding concept in the group-case.

Theorem 1. *Let K be a commutative hypergroup, which is a group. The continuous function $\varphi : K \to \mathbb{C}$ is a generalized exponential monomial in the hypergroup-sense if and only if it is a generalized exponential monomial in the group-sense. In particular, the continuous function $\varphi : K \to \mathbb{C}$ is an exponential monomial in the hypergroup-sense if and only if it is an exponential monomial in the group-sense.*

Proof. We write $*$ as $+$ in K. Suppose that $\varphi : K \to \mathbb{C}$ is continuous and satisfies (5) with some exponential $m : K \to \mathbb{C}$ for all $y_1, y_2, \dots, y_{n+1}$ in K.

It is easy to check that it follows

(6)
$$m(x+y_1+y_2+\cdots+y_{n+1})\Delta_{y_1,y_2,\ldots,y_{n+1}}(\varphi\cdot\check{m})(x)=\Delta_{m;y_1,y_2,\ldots,y_{n+1}}\varphi(x)=0$$

for each $x,y_1,y_2,\ldots,y_{n+1}$ in K. As m is never zero, this implies

$$\Delta_{y_1,y_2,\ldots,y_{n+1}}(\varphi\cdot\check{m})(x)=0$$

for each $x,y_1,y_2,\ldots,y_{n+1}$ in K, hence $\varphi\cdot\check{m}$ is a generalized polynomial and our statement follows.

The converse statement follows similarly. Indeed, if φ is a generalized exponential monomial of the form $\varphi=p\,m$ with a generalized polynomial p and an exponential m, then the function $\varphi\cdot\check{m}$ is a generalized polynomial satisfying (4), which is, by (6), equivalent to (5). The theorem is proved. □

As an application of this concept we prove a simple result.

Theorem 2. *Let K be a commutative hypergroup and $(\varphi_n)_{n\in\mathbb{N}}$ a generalized moment function sequence. Then φ_n is an exponential monomial of degree at most n corresponding to the exponential φ_0 for each n.*

Proof. By definition the sequence satisfies

$$\varphi_n(x*y)=\sum_{k=0}^{n}\varphi_k(x)\varphi_{n-k}(y)$$

for each x,y in K $(n=0,1,\ldots)$. We prove the statement by induction on n and it is obvious for $n=0$. Clearly $m=\varphi_0$ is an exponential. Suppose that $n\geqslant 1$ and we have proved our statement for $k\leqslant n-1$. Now we prove it for $k=n$. Let $y_1,y_2,\ldots,y_{n+1}$ be arbitrary in K. We have

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}}\varphi_n(x)=\Delta_{m;y_1,y_2,\ldots,y_n}\big[\varphi_n(x*y_{n+1})-m(y_{n+1})\varphi_n(x)\big]=$$

$$\Delta_{m;y_1,y_2,\ldots,y_n}\Big[\sum_{k=0}^{n}\binom{n}{k}\varphi_k(x)\varphi_{n-k}(y_{n+1})\Big]-m(y_{n+1})\Delta_{m;y_1,y_2,\ldots,y_n}\varphi_n(x)=$$

$$\Delta_{m;y_1,y_2,\ldots,y_n}\varphi_n(x)\cdot m(y_{n+1})-m(y_{n+1})\Delta_{m;y_1,y_2,\ldots,y_{n+1}}\varphi_n(x)=0\,,$$

which proves our statement. □

We prove our main results, which show that this concept is a generalization of the one has been used in [5, 7, 8, 9]. We recall that given a commutative hypergroup K and a positive integer n the function $\Phi:K\times\mathbb{C}^n\to\mathbb{C}$ is called an *exponential family* (see e.g. [8]), if

1. for each x in K the function $\lambda\mapsto\Phi(x,\lambda)$ is $\mathcal{C}^\infty$ on $\mathbb{C}^n$;
2. for each λ in $\mathbb{C}^n$ the function $x\mapsto\Phi(x,\lambda)$ is an exponential on K;
3. for each exponential m on K there is a λ in $\mathbb{C}^n$ for which $m(x)=\Phi(x,\lambda)$ holds, whenever x is in K.

Lemma 3. *Let K be a commutative hypergroup, $\Phi : K \times \mathbb{C}^n \to \mathbb{C}$ an exponential family for K, N a natural number, λ a complex number and $P : \mathbb{C}^n \to \mathbb{C}$ a polynomial of degree N. Then the function $\varphi : x \mapsto P(\partial_\lambda)\Phi(x, \lambda)$ is an exponential monomial of degree at most N corresponding to the exponential $m : x \mapsto \Phi(x, \lambda)$.*

Proof. We have to show that

$$\Delta_{m;y_1,y_2,\dots,y_{N+1}}\varphi(x) = 0$$

holds for each $y_1, y_2, \dots, y_{N+1}$ in K. Obviously it is enough to prove the statement for polynomials of the form $P(\xi) = \xi^\alpha$, where α is a multi-index in $\mathbb{N}^n$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2} \cdot \dots \cdot \xi_n^{\alpha_n}$. We prove by induction on $|\alpha|$ and the statement is obviously true for $|\alpha| = 0$. We have for each x, y in K

$$(7)\quad \partial_1^{\alpha_1}\partial_2^{\alpha_2} \cdot \dots \cdot \partial_n^{\alpha_n}\big(\Phi(x,\lambda)\cdot\Phi(y,\lambda)\big) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta\Phi(x,\lambda)\partial_\lambda^{\alpha-\beta}\Phi(y,\lambda) =$$

$$\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta\Phi(x,\lambda)\partial_\lambda^{\alpha-\beta}\Phi(y,\lambda) + P(\partial_\lambda)\Phi(y,\lambda)\,.$$

With the notation $l = |\alpha|$ it follows from (7)

$$\Delta_{m;y_1,y_2,\dots,y_{l+1}}\partial_\lambda^\alpha\Phi(x,\lambda) = \Delta_{m;y_1,y_2,\dots,y_l}\big[\Delta_{m;y_{l+1}}\partial_\lambda^\alpha\Phi(x,\lambda)\big] =$$

$$\Delta_{m;y_1,y_2,\dots,y_l}\big[\partial_\lambda^\alpha\Phi(x*y_{l+1},\lambda) - \partial_\lambda^\alpha\Phi(y_{l+1},\lambda)\Phi(x,\lambda)\big] =$$

$$\Delta_{m;y_1,y_2,\dots,y_l}\big[\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial_\lambda^\beta\Phi(x,\lambda)\partial_\lambda^{\alpha-\beta}\Phi(y_{l+1},\lambda)\big] =$$

$$\sum_{\beta<\alpha}\binom{\alpha}{\beta}\big[\Delta_{m;y_1,y_2,\dots,y_l}\partial_\lambda^\beta\Phi(x,\lambda)\big]\partial_\lambda^{\alpha-\beta}\Phi(y_{l+1},\lambda) = 0\,,$$

by assumption, as for each $\beta < \alpha$ the function $x \mapsto \partial_\lambda^\beta\Phi(x,\lambda)$ is an exponential monomial of degree at most $|\alpha| - 1 < l$. This means that the function $x \mapsto P(\partial_\lambda)\Phi(x,\lambda)$ is a generalized exponential monomial of degree at most l and equation (7) implies that it generates a finite dimensional variety, hence it is actually an exponential monomial. □

We can formulate the following consequences.

Corollary 4. *Let K be a polynomial hypergroup generated by the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. Then for each complex number λ and natural number k the functions $n \mapsto P_n^{(k)}(\lambda)$ are exponential monomials of degree at most k on K.*

Proof. The statement follows from Theorem 3, as $(n,\lambda) \mapsto P_n(\lambda)$ is an exponential family on K, by Theorem 2.2., p. 40. in [8]. □

In the following statement we use the notation $\mathbb{R}_0$ for the set of all nonnegative real numbers.

Corollary 5. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function $A : \mathbb{R}_0 \to \mathbb{R}$, further for each complex number λ let $\Phi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ be the unique solution of the initial value problem*

$$\frac{d^2}{dx^2}\Phi(x,\lambda) + \frac{A'(x)}{A(x)}\frac{d}{dx}\Phi(x,\lambda) = \lambda\Phi(x,\lambda) \text{ for } x > 0\,, \tag{8}$$

$$\Phi(0,\lambda) = 1\,, \tag{9}$$

$$\frac{d}{dx}\Phi(0,\lambda) = 0\,. \tag{10}$$

Then for every complex number λ and for each natural number k the functions $x \mapsto \frac{d^k}{d\lambda^k}\Phi(x,\lambda)$ are exponential monomials of degree at most k on K.

Proof. The statement follows from Theorem 3, as $(x,\lambda) \mapsto \Phi(x,\lambda)$ is an exponential family on K, by Theorem 4.2., p. 62. in [8]. □

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