Internat. J. Functional Analysis, Operator Theory and Applications
© 2013 Pushpa Publishing House, Allahabad, India
Published Online: October 2013
Available online at http://pphmj.com/journals/ijaota.htm
Volume 5, Number 2, 2013, Pages 119-139

UNIFIED APPROACH TO VECTOR VALUED INTEGRATION

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Abstract

We introduce a natural and more flexible approach to the definition of vector valued integral that will completely forgo any measurability assumption, strengthen the existing various classical concepts of integral, and provide a continuous thread tying the subject matter together. As applications, we obtain extensions of the Lebesgue convergence theorems, the Dvoretsky-Rogers theorem, and the Orlicz-Pettis theorem.

1. Introduction

Thanks to its close parallelisms with the numerical Lebesgue integral, the Bochner integral continues to be the reference whenever it comes to vector integration (see, for example, [3]) and proves to be still a very powerful tool in many mathematical applications. However, its definition is not simple due to the requirement of measure theory. On the other hand, the extension to vector integration of the Henstock-Kurzweil gauge integral technique, which

Received: May 21, 2013; Accepted: June 30, 2013

2010 Mathematics Subject Classification: Primary 46G10; Secondary 46E40, 46B22.

Keywords and phrases: vector valued Henstock-Kurzweil integral, Lebesgue-Bochner integral.

has been of interest to several authors [4, 8, 7, 12], since Henstock [9] launched the idea in 1969, seems to offer a simpler and more transparent alternative. However, its definition is for the most part limited to compact subsets of metric spaces. In an attempt to remedy the limitation of the Henstock-Kurzweil integral, and at the same time aiming to unify the two approaches, we have proposed in [11] a construction that allows us to define vector valued Henstock-Kurzweil integration on any abstract measure space.

However, the assumption of measurability is arguably still too heavy and rather restrictive in some cases in many important applications, especially when the range space is not separable. For instance, the function $t \mapsto \delta_t$ that associates to every $t \in [a, b]$, the Dirac measure δ_t in the space $C[a, b]^*$ of Radon measures on $[a, b]$ is known to be non-measurable, however, $t \mapsto \| \delta_t \|$ is measurable and $\int_{[a, b]} \| \delta_t \| dt < \infty$. Other examples of non-measurable operator-valued functions $t \mapsto U_t$ are also given in [12] for which $\int \| U_t \| dt < \infty$.

The purpose of the present paper is to provide an intuitive approach to integration that will do without any measurability assumption. We closely follow the same construction as in [11]. The only major difference is that we replace “measure” with a more general notion of “size function”. We use net limit which is a generalization of the notion of sequential limit (see, for example, [10]) introduced by Moore and Smith, and which is undoubtedly the most adapted approximation technique for the definition of the integral. We obtain a relatively simple definition of the integral that not only extends but also unifies the existing various notions of integral. This treatment allows us to bypass the seemingly unavoidable notion of measure theory of the Lebesgue-Bochner integral.

We present our definition and some immediate basic properties of the integral in Section 2. Section 3 is devoted to the study of different classes of integrable functions. We study the vector valued extension of the Lebesgue integral. We obtain generalizations of the Lebesgue convergence theorems.

In Section 4, we discuss further notions of integrability; we obtain results concerning unconditional, norm and weak integrability of functions, extending the Dvoretsky-Rogers theorem and the Orlicz-Pettis theorem.

2. Definition of the Integral

In this section, we lay out the essential elements for the definition of the integral. For the range space, we always consider a linear vector space X equipped with a norm $\|\cdot\|$. An approximation technique must be at hand in order to define any suitable integral.

A nonempty set D is said to be *directed* by a binary relation $\succ$, if $\succ$ has the following properties:

(1) if $x, y, z \in D$ such that $x \succ y$ and $y \succ z$, then $x \succ z$ (transitivity);

(2) if $x, y \in D$, then there exists $z \in D$ such that $z \succ x$ and $z \succ y$ (upper bound property).

Given a set X, a net of elements of X is an X-valued function defined on a directed set $(D, \succ)$. The notion of convergence can be defined whenever the set X is a metric space, with distance function d.

A net $f : (D, \succ) \to (X, d)$ is said to be *convergent* if there exists an element $x \in X$ such that for every $\varepsilon > 0$, there exists $\omega_0 \in D$ such that for every $\omega \succ \omega_0$, $d(f(\omega), x) < \varepsilon$.

Ordinary sequences $n \mapsto a_n$, in which $D = \mathbb{N}$ directed by $>$, constitute a special case of net. The net limit takes over all the essential parts of the theory of limits of sequences; to name a few: the uniqueness of limit, the algebraic properties of limits, the Cauchy criterion, and so on. For more details on net limits, we refer the reader to [10].

The next essential element for the definition of the integral is the "sizing" of the subsets of the domain space. In the following, Ω is a nonempty set, the power set of Ω, that is, the set of all subsets of Ω is denoted by 2^{Ω}.

By a size function, we mean a set function $\ell : 2^{\Omega} \to [0, +\infty]$ that satisfies the following conditions:

- $\ell(\varnothing) = 0$;
- $\ell(A) \le \ell(B)$ whenever $A \subset B$; (monotonicity);
- $\ell\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \ell(A_n)$ (countable subadditivity).

The Carathéodory extensions of measures provide us with ample examples of such size functions.

If $A \in 2^{\Omega}$, then we call *subpartition* of A any collection $P = \{I_1, I_2, ..., I_n\}$ of finitely many pairwise disjoint subsets of A of finite size; that is, $I_i \cap I_j = \varnothing$ whenever $i \ne j$ and $\ell(I_i) < \infty$ for all $i \in \{1, ..., n\}$. We denote $\sqcup P$ the subset of A obtained by taking the union of all elements of P. A partition $P = \{I_i : i = 1, ..., n\}$ is said to be *tagged* if a point $t_i \in I_i$ is chosen for each $i \in \{1, ..., n\}$. We write $P := \{(I_i, t_i) : i \in \{1, ..., n\}\}$ if we wish to specify the tagging points. We denote by $\Pi(A)$ the collection of all tagged subpartitions of the set A. The *mesh* or the *norm* of $P \in \Pi(A)$ is defined to be

$$\| P \| = \max\{\ell(I_i) : I_i \in P\}.$$

If $P, Q \in \Pi(A)$, then we say that Q is a *refinement* of P and we write $Q \succ P$ if $\| Q \| \le \| P \|$ and $\sqcup P \subset \sqcup Q$. Clearly, such a relation does not depend on the tagging. It is readily seen that the relation $\succ$ is transitive on $\Pi(A)$. If $P, Q \in \Pi(A)$, then we denote by

$$P \vee Q := \{I \backslash J,\ I \cap J,\ J \backslash I : I \in P,\ J \in Q\}.$$

It is readily seen that $P \vee Q \in \Pi(A)$ and that $P \vee Q \succ P$ and $P \vee Q \succ Q$. Thus, the relation $\succ$ has the upper bound property on $\Pi(A)$. We infer that the set $\Pi(A)$ is directed by the binary relation $\succ$.

Given a function $f : \Omega \to X$, and a tagged partition $P := \{(I_i, t_i) : i \in \{1, ..., n\}\} \in \Pi(A)$, we define the *Riemann sum* to be

$$\ell_f(P) = \sum_{i=1}^{n} \ell(I_i) f(t_i).$$

Thus, the function $P \mapsto \ell_f(P)$ is an X-valued net on the directed set $\Pi(A)$. For convenience, we are going to denote $\int_A f d\ell := \lim_{\succ} \ell_f(P)$ whether or not the net limit exists.

We now hold all the necessary tools to define the notion of integrability.

Definition 1. We say that a function $f : \Omega \to X$ is (*strongly*) *integrable* over a set $A \subset \Omega$ with respect to a given size function ℓ (or ℓ*-integrable* for short) if $\int_A f d\ell$ represents a vector in X. The vector $\int_A f d\ell$ is then called the ℓ*-integral* of f over the set A.

In other words, $f : \Omega \to X$ is ℓ-integrable over the set A with ℓ-integral $\int_A f d\ell$ if for every $\varepsilon > 0$, there exists $P_0 \in \Pi(A)$ such that for every $P \in \Pi(A)$, $P \succ P_0$, we have

$$\left\| \ell_f(P) - \int_A f d\ell \right\| < \varepsilon. \tag{2.1}$$

We shall denote by $\mathcal{I}(A, \ell, X)$ the set of all ℓ-integrable functions over the set A.

It now becomes transparent that indeed our approach does not require any knowledge of measure theory. It also has the advantage that many classical properties of the integral follow immediately from the properties of net limits and therefore their proofs are obtained at no cost at all.

For example, the uniqueness of net limits ensures us that there exists at most one vector $\int_A f d\ell$ that satisfies the property in Definition 1. We also immediately infer that being a limit operator, the integral is linear, and that $\mathcal{I}(A, \ell, X)$ is indeed a vector space.

It is clear that if A and B are disjoint subsets of Ω, then every subpartition R of $A \sqcup B$ is of the form $P \sqcup Q$, where $P \in \Pi(A)$ and $Q \in \Pi(B)$. It then follows that $\ell_f(R) = \ell_f(P) + \ell_f(Q)$. Thus,

Proposition 2. *If a function* $f : \Omega \to X$ *is* ℓ*-integrable over both a set A and a set B, then f is* ℓ*-integrable over* $A \sqcup B$ *and*

$$\int_{A \sqcup B} f d\ell = \int_A f d\ell + \int_B f d\ell.$$

Now assume that the range space X has a lattice structure compatible with its norm. The following is known as the monotonicity property of the integral.

Proposition 3. *Let* $f, g : \Omega \to X$ *be both* ℓ*-integrable over A, and* $h : \Omega \to X$ *such that* $f(\omega) \leq h(\omega) \leq g(\omega)$, *for all* $\omega \in A$, *then*

(1) *h is* ℓ*-integrable, and*

(2) $\int_A f d\ell \leq \int_A h d\ell \leq \int_A g d\ell.$

Proof. It suffices to notice that for all $P \in \Pi(A)$, one has $\ell_f(P) \leq \ell_h(P) \leq \ell_g(P)$. □

It is a well known fact that, if X is a complete space, then a net of element if X is convergent if and only if it satisfies the Cauchy criterion (see, for example, [10]). It follows that

Proposition 4. *Let X be a Banach space. Then a function* $f : \Omega \to X$ *is* ℓ*-integrable if and only if for every* $\varepsilon > 0$, *there exists* $P_0 \in \Pi(A)$ *such that for every* $P, Q \in \Pi(A)$, $P, Q \succ P_0$, *we have*

$$\| \ell_f(P) - \ell_f(Q) \| < \varepsilon. \tag{2.2}$$

We next see how Definition 1 provides an alternative generalization that permits integration over any arbitrary subset of Ω. It also broadens the class of integrable functions 1 large enough to contain most of the classical notions of integral.

Let $(I)_{i=1}^{n}$ be a finite partition of the interval $[a, b]$, $c_i \in I_i$ for each i, and let δ be a positive function on $[a, b]$. The collection of pairs $(I_i, c_i)_{i=1}^{n}$ is said to be δ-*fine* if for each i, $c_i \in I_i \subset (c_i - \delta(c_i), c_i + \delta(c_i))$. A function $f : [a, b] \to X$ is said to be *Henstock-Kurzweil integrable* on $[a, b]$ if there exists an element $\int_{[a,b]}^{HK} f \in X$ with the property: for $\varepsilon > 0$, there is a function $\delta : [a, b] \to [0, \infty)$ such that the inequality

$$\left\| \sum_{i=1}^{n} f(c_i)\ell(I_i) - \int_{[a,b]}^{HK} fd\ell \right\| < \varepsilon$$

holds for every δ-fine partition $(I_i, c_i)_{i=1}^{n}$. Here $\ell(I_i) = b_i - a_i$ is the natural length of the interval I_i. Let K be the subset of $2^{\mathbb{R}}$ consisting of compact intervals. For each subset of A of $\mathbb{R}$, if there exists $\{I_n\} \subset K$ such that $A \subset \bigcup_n I_n$, then we define

$$\ell^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subset \bigcup_n I_n, I_n \in K \right\},$$

otherwise, we let $\ell^*(A) = \infty$. It is readily seen that ℓ^* is a size function. We notice that δ-fine partitions are elements of $\Pi([a, b])$, and any refinement of a δ-fine partition is a δ-fine partition, and

$$\sum_{i=1}^{n} f(c_i)\ell(I_i) = \ell^*{}_f(P),$$

where $P = \{(I_i, c_i) : i = 1, ..., n\}$. We infer that Henstock-Kurzweil integrable functions are ℓ^*-integrable in the sense of our Definition 1, and the integrals do coincide. The book of Bartle [1] is a good source on scalar-valued Henstock-Kurzweil integral. For the vector valued case, the reader is referred to [4, 6].

Small changes can be made to extend the Henstock-Kurzweil integral to any arbitrary subset of a set Ω on which a size function ℓ is defined. We first extend the notion of δ-fine as partition follows:

Given a positive function δ *on* Ω, *we say that a subpartition* $P := \{(I_i, t_i) : i \in \{1, \ldots, n\}\} \in \Pi(A)$ *is* ℓ-δ-*fine if for* $i = 1, \ldots, n$,

$$\ell(I_i) \leq 2\delta(t_i).$$

Definition 5. A function $f : \Omega \to X$ is said to be (*generalized*) *Henstock-Kurzweil* ℓ*-integrable* over a set $A \subset \Omega$, if there exists an element $\int_A^{HK} f \in X$ such that for $\varepsilon > 0$ there is a function $\delta : \Omega \to [0, \infty)$ such that the inequality

$$\left\| \ell_f(P) - \int_A^{HK} f d\ell \right\| < \varepsilon$$

holds for every ℓ-δ-fine subpartition $P \in \Pi(A)$. We denote by $\mathcal{HK}(A, \ell, X)$ the set of all the generalized Henstock-Kurzweil ℓ-integrable functions over the set *A*.

It follows from our discussion above that

Proposition 6. $\mathcal{HK}(A, \ell, X) \subset \mathcal{I}(A, \ell, X)$ *and* $\int_A^{HK} f d\ell = \int_A f d\ell$ *for every* $f \in \mathcal{HK}(A, \ell, X)$.

Now, let (Ω, Σ, μ) be a measure space, that is to say, Σ is a σ-algebra, μ is a measure defined on Σ. The elements of Σ are called *measurable sets*. If *A* is a measurable set, then we denote by Σ_A the subsets of Σ consisting of measurable subsets of *A*. A subpartition $P \in \Pi(A)$ is said to be *measurable* if the elements of P are taken from Σ_A. We denote by $\Pi(A, \mu)$ the subset of $\Pi(A)$ consisting of measurable subpartitions of A. $\Pi(A, \mu)$ is clearly seen to be stable under refinement. The outer-measure μ^* associated to the measure μ, that is, the Carathéodory extension of μ is clearly a size function on 2^n. Given a σ-algebra Σ' of subsets of Ω, a function $f : \Omega \to \mathbb{R}$ is Σ'-measurable if $f^{-1}(U) \in \Sigma'$ whenever U is an open subset of $\mathbb{R}$. Since the restriction of μ^* to Σ coincides with μ, it follows that check that a measurable function is Lebesgue integrable with respect to a measure μ if and only if both f and $|f|$ are μ^*-integrable in the sense of Definition 1 and the integrals do coincide. A classical reference for details on Lebesgue integral is the book of Bartle and Sherbert [2].

The vector valued extension of the Lebesgue integral will be treated in the next section.

3. Space of Integrable Functions

In what follows, X is a vector space with norm $\|\cdot\|$. We have already noticed that the space $\mathcal{I}(A, \ell, X)$ has a vector space structure. In this section, we briefly show that if X is a Banach space and $\ell(A) < \infty$, then $\mathcal{I}(A, \ell, X)$ can be given the structure of a complete seminormed space. We also introduce and study the notion of norm-integrability. We obtain generalizations of the convergence theorems of the Lebesgue integral.

Definition 7. For every $f \in \mathcal{I}(A, \ell, X)$, we define the $\Pi(A)$-*variation* of f to be

$$\| f \|_{\Pi(A)} := \sup\{\| \ell_f(P) \| : P \in \Pi(A)\}.$$

We say that the function f is of *bounded* $\Pi(A)$-*variation* if $\| f \|_{\Pi(A)} < \infty$. The collection of all functions of bounded $\Pi(A)$-variation will be denoted by $\Pi(A, \ell, X)$.

It is clear that $\mathcal{I}(A, \ell, X) \subset \Pi(A, \ell, X)$. It is readily seen that $f \mapsto \| f \|_{\Pi(A)}$ defines a seminorm on $\Pi(A, \ell, X)$. For the particular case where $\ell(A)$ is finite, we have the following theorem, the proof of which goes exactly in the same way as in the proof of Theorem 3 of [11].

Theorem 8. *Let $\ell : 2^{\Omega} \to [0, \infty]$ be a size function, $A \in 2^{\Omega}$ be such that $\ell(A) < \infty$, and let X be a Banach space. Then the function space $\mathcal{I}(A, \ell, X)$ is complete with respect to the seminorm $\| \cdot \|_{\Pi(A)}$.*

It should be also clear that if the set *A* is such that $\ell(A) = 0$, then for all subpartitions $P \in \Pi(A)$, $\ell_f(P) = 0$, and thus $\int_A fd\ell = 0$. It follows that the integral does not distinguish between functions which differ only on set of size zero. To make this more precise,

$$\int_A fd\ell = \int_A gd\ell \text{ whenever } \ell\{x \in A : f(x) \neq g(x)\} = 0.$$

We say that a function *f* is essentially equal on *A* to another function *g*, and we write $f \sim g$ if $\ell\{x \in A : f(x) \neq g(x)\} = 0$. It is readily seen that the relation $f \sim g$ is an equivalence relation on $\mathcal{I}(A, \ell, X)$. We shall denote by $I(A, \ell, X)$ the quotient space $\mathcal{I}(A, \ell, X)/\sim$. The restriction of the seminorm $\| \cdot \|_{\Pi(A)}$ is a norm on $I(A, \ell, X)$. As a corollary of Theorem 8, we get

Theorem 9. *Let $\ell : 2^{\Omega} \to [0, \infty]$ be a size function, $A \in 2^{\Omega}$ be such that $\ell(A) < \infty$, and let X be a Banach space. Then $I(A, \ell, X)$ is Banach space with respect to the norm $\| \cdot \|_{\Pi(A)}$.*

Definition 10. We say that a function $f : \Omega \to X$ is *norm-integrable* over a set $A \subset \Omega$ with respect to a given size function ℓ if the numerical function

$$\| f(\cdot) \| : \Omega \to [0, \infty) : \omega \mapsto \| f(\omega) \|$$

is ℓ-integrable over the set *A*. We denote by $\| \mathcal{I} \|(A, \ell, X)$ the space of all norm-integrable functions over the set *A*.

We observe that $\|\cdot\|_1 : f \mapsto \int_A \| f(\cdot)\| d\ell$ defines a seminorm on $\|\mathcal{I}\|(A, \ell, X)$, and it is easily checked that if X is a Banach space $\|\mathcal{I}\|(A, \ell, X)$ is complete when endowed with the seminorm $\|\cdot\|_1$. Hence, the space $\| I \|(A, \ell, X) = \|\mathcal{I}\|(A, \ell, X)/\sim$ is a Banach space with the norm $\|\cdot\|_1$.

One of the main reasons behind the Lebesgue-Bochner integral's power and popularity is the convergence theorems that apply to it. We shall see next that our extension of the definition of integral yields a space that already possesses those remarkable properties.

Fatou's Lemma.

Theorem 11. *Let* $f_n : \Omega \to X$ *be a sequence of functions satisfying for every* $\omega \in A$, $f(\omega) := \liminf_{n\to\infty} \| f_n(\omega)\|$. *Then*

$$\int_A f d\ell \leq \liminf_{n\to\infty} \int_A \| f_n(\cdot)\| d\ell.$$

Proof. By the definition of the integral, it is enough to show that

$$\ell_f(P) \leq \liminf_{n\to\infty} \int_A \| f_n(\cdot)\| d\ell$$

for all $P = \{(I_i, t_i) : i = 1, \ldots, m\} \in \Pi(A)$. Fix $P \in \Pi(A)$. Define $\varphi_P(\omega) = \sum_{I_i \in P} 1_I(\omega) f(t_i)$. We notice that $\ell_{\varphi_P}(P) = \ell_f(P)$. We let $a = \min\{\varphi_P(\omega) : \omega \in A\}$, $M = \max\{\varphi_P(\omega) : \omega \in A\}$, and we define

$$E = \{\omega \in A : \varphi_P(\omega) > a\}.$$

We first assume that $\ell_{\varphi_P}(P) = \infty$. Then $\ell_{\varphi_P}(P) \leq M\ell(E)$. Hence $\ell(E) = \infty$. Next, we define

$$E_n = \{\omega \in A : \| f_k(\omega)\| > a, \forall k \geq n\}.$$

Then $E \subset \bigcup_n E_n$, and $E_n \subset E_{n+1}$ for all n. Thus, $\ell(\bigcup_n E_n) = \infty$, and

$\ell\left(\bigcup_{k=1}^{n} E_k\right) \leq \ell(E_{n+1})$. It follows that $\lim_{n\to\infty} \ell(E_n) = \infty$. On the other hand, we have $\int_A \| f_n(\cdot)\| d\ell \geq a\ell(E_n)$. It follows that

$$\ell_f(P) = \infty = \liminf_{n\to\infty} \int_A \| f_n(\cdot)\| d\ell.$$

We now look at the case $\ell_{\varphi_P}(P) < \infty$. Then $a\ell(E) \leq \ell_{\varphi_P}(P)$ and thus $\ell(E) < \infty$. Fix $\varepsilon > 0$ and define

$$E_n = \{\omega \in A : \| f_k(\omega)\| > (1-\varepsilon)\varphi_P(\omega),\ \forall k \geq n\}.$$

Then $E \subset \bigcup_n E_n$, and $E_n \subset E_{n+1}$ for all n. Since $\ell(E) < \infty$, we have

$$\lim_{n\to\infty} \ell(E \backslash E_n) = \ell(\varnothing) = 0.$$

Thus, we can choose an integer n_0 such that $\ell(E\backslash E_n) < \varepsilon$ for all $n > n_0$. Thus, if $n > n_0$, then we have

$$\int_A \| f_n(\cdot)\| d\ell \geq \int_{E_n} \| f_n(\cdot)\| d\ell \geq (1-\varepsilon)\int_{E_n} \varphi_P(\omega) d\ell.$$

On the other hand,

$$\int_A \varphi_P d\ell = \int_E \varphi_P d\ell = \int_{E_n} \varphi_P d\ell + \int_{E\backslash E_n} \varphi_P d\ell.$$

Hence

$$\begin{aligned}
\int_A \| f_n(\cdot)\| d\ell &\geq (1-\varepsilon)\int_{E_n} \varphi_P(\omega) d\ell \\
&= (1-\varepsilon)\left[\int_A \varphi_P d\ell - \int_{E\backslash E_n} \varphi_P d\ell\right] \\
&\geq (1-\varepsilon)\left[\int_A \varphi_P d\ell - \varepsilon M\right] \\
&= \int_A \varphi_P d\ell - \varepsilon\left[\int_A \varphi_P d\ell + \varepsilon M\right].
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\liminf_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell \geq \int_A \varphi_P d\ell = \ell_f(P).$$

The proof is complete. □

Monotone Convergence Theorem.

Theorem 12. *Let* $f_n : \Omega \to X$ *be a sequence of functions satisfying*:

(1) $0 \leq \| f_n(\omega) \| \leq \| f_{n+1}(\omega) \|$, *for every* $\omega \in A \subset \Omega$ *and for all* $n \in \mathbb{N}$;

(2) *for every* $\omega \in A$, $f(\omega) := \lim_{n\to\infty} f_n(\omega)$.

Then $f : A \to [0, \infty]$ *is* ℓ*-norm-integrable if and only if*

$$\lim_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell < \infty.$$

Moreover,

$$\int_A \| f(\cdot) \| d\ell = \lim_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell.$$

Proof. It follows from the monotonicity property of the integral (Proposition 3) that

$$\int_A \| f_n(\cdot) \| d\ell \leq \int_A \| f_{n+1}(\cdot) \| d\ell \leq \int_A \| f(\cdot) \| d\ell.$$

Hence the sequence $n \mapsto \int_A \| f_n(\cdot) \| d\ell$ is a non-decreasing sequence of real numbers. On the one hand, if f is integrable, that is, if $\int_A \| f(\cdot) \| d\ell < \infty$, then

$$\lim_{n\to\infty} \int_A \| f(\cdot) \| d\ell \leq \int_A \| f(\cdot) \| d\ell < \infty.$$

On the other hand, if $\lim_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell < \infty$, then by Fatou's Lemma, we have

$$\int_A \| f(\cdot) \| d\ell \le \liminf_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell = \lim_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell.$$

In both the cases, we have $\lim_{n\to\infty} \int_A \| f_n(\cdot) \| d\ell = \int_A \| f(\cdot) \| d\ell.$ □

Dominated Convergence Theorem.

Theorem 13. *Let X be a lattice normed space. Let* $f_n : \Omega \to X$ *be a sequence of functions satisfying the following properties*:

(1) $f_n(\omega) \to f(\omega)$ *for all* $\omega \in A \subset \Omega$;

(2) *there exists a real valued function h* ℓ*-integrable over the set A such that* $\| f_n(\omega) \| \le h(\omega)$ *for all* $\omega \in A$, *and for all* $n \in \mathbb{N}$.

Then

(1) *f is* ℓ*-integrable*;

(2) $\lim_{n\to\infty} \int_A \| f - f_n \| d\ell = 0$;

(3) $\int_A f d\ell = \lim_{n\to\infty} \int_A f_n d\ell.$

Proof. It follows from the conditions of the theorem that for all $\omega \in A$, and for all $n \in \mathbb{N}$, we have

$$\| f(\omega) - f_n(\omega) \| \le 2h(\omega)$$

and $\limsup_{n\to\infty} \| f(\omega) - f_n(\omega) \| = 0$ for each $\omega \in A$. Using linearity and monotonicity of the integral, we get that

$$\left\| \int_A f d\ell - \int_A f_n d\ell \right\| = \left\| \int_A (f - f_n) d\ell \right\| \le \int_A \| f - f_n \| d\ell.$$

By Fatou's Lemma, we have

$$\limsup_{n\to\infty} \int_A \| f - f_n \| d\ell \le \int_A \limsup_{n\to\infty} \| f - f_n \| d\ell = 0.$$

The result follows. □

As a direct consequence of the Dominated Convergence Theorem 13, we obtain that norm-integrable functions are integrable, i.e., $\|\mathcal{I}\|(A, \ell, X) \subset \mathcal{I}(A, \ell, X)$. The reverse inclusion does not hold as evidenced by the classical example of the real valued function $f(x) = \frac{\sin x}{x}$ defined on $[0, \infty)$.

We note that if the size function $\ell = \mu^*$ is the Carathéodory extension of some measure μ defined on a $\Sigma\,\sigma$-algebra of subsets of A, then norm-integrability and Bochner-integrability agree for strongly μ-measurable functions.

4. Further Notions of Integrability

An analogue definition can also be introduced if we replace in Definition 5 $\Pi(A)$ by $\tilde{\Pi}(A)$, the set of all tagged partitions $P := \{(I_i, t_i) : i \in \{1, \ldots, n\}\}$ for which each tagging t_i is in $\sqcup P$ but is not required to be in I_i for each $i = 1, \ldots, n$.

Definition 14. A function $f : \Omega \to X$ is said to be *unconditionally* ℓ*-integrable* over a set $A \subset \Omega$, if there exists an element $\tilde{\int_A} f \in X$ such that for $\varepsilon > 0$ there is a function $\delta : \Omega \to [0, \infty)$ such that the inequality

$$\left\| \ell_f(P) - \tilde{\int_A} fd\ell \right\| < \varepsilon$$

holds for every ℓ-δ-fine subpartition $P \in \tilde{\Pi}(A)$. We denote by $\tilde{\mathcal{I}}(A, \ell, X)$ the set of all unconditionally ℓ-integrable functions over the set A.

It becomes clear as in the above discussion that

Proposition 15. *$\tilde{\mathcal{I}}(A, \ell, X) \subset \mathcal{I}(A, \ell, X)$ and $\tilde{\int_A} fd\ell = \int_A fd\ell$ for every $f \in \tilde{\mathcal{I}}(A, \ell, X)$.*

We note that if the size function $\ell = \mu^*$ for some measure μ defined on a σ-algebra containing the Borel subsets of the set *A*, then Definition 14 is readily seen as an extension of the McShane integrability with respect to the measure μ.

For finite dimensional Euclidean spaces *X*, it is easily checked that norm and unconditional ℓ-integrability coincide, that is, $\widetilde{\mathcal{I}}(A, \ell, X) = \|\mathcal{I}\|(A, \ell, X)$. The situation is different for infinite dimensional spaces. The well known Dvoretsky-Rogers Theorem (see, for example, [3]) asserts that in an infinite dimensional Banach space, there always exists an unconditional but not norm-summable sequence. Our next result can be seen as an extension of such a result.

Theorem 16. *Let X be an infinite dimensional Banach space. Let* $\ell = \mu^*$ *for some regular measure* μ *defined on a* σ*-algebra containing the Borel subsets of* Ω*. Let* $A \in 2^{\Omega}$ *be compact. Then there exists a function* $f : \Omega \to X$ *which is unconditionally* ℓ*-integrable but not norm-*ℓ*-integrable over the set A.*

Proof. Since *X* is infinite dimensional, there exists a sequence $\{x_n\}$ of elements of *X* that is unconditionally summable but not norm summable. Let $\{U_n\}$ be a pairwise disjoint sequence of open subsets of *A*. Then, since ℓ is σ-additive on Borel subsets of *A*, we have

$$\sum_{n=1}^{\infty} \ell(U_n) \le \ell(A) < \infty.$$

By removing sets of size 0, we may assume that $\ell(U_n) > 0$ for all *n*. Set $y_n = [\ell(U_n)]^{-1} x_n$ for each $n \in \mathbb{N}$. Then, by our hypothesis, the series $\sum_n \ell(U_n) y_n$ is unconditionally convergent, say to *x*, while $\sum_n \ell(U_n) \| y_n \| = \infty$. Define

$$f(t) = \begin{cases} y_n & \text{if } t \in U_n,\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int \| f(\cdot) \| d\ell = \sum_n \ell(U_n) \| y_n \| = \infty$, that is, $f \notin \| \mathcal{I} \| (A, \ell, X)$. On the other hand, given $\varepsilon > 0$, there exists a finite subset $N_\varepsilon \subset \mathbb{N}$ such that

$$\left\| \sum_{n \leq K \bigcup N_\varepsilon} \ell(U_n) y_n - x \right\| < \varepsilon \tag{4.1}$$

for every finite subset K of $\mathbb{N}$. By the Cauchy criterion, there exists $N_0 \geq \max N_\varepsilon$ such that for every finite subset N of $\mathbb{N} \setminus \{1, 2, ..., N_0\}$, we have

$$\left\| \sum_{n \in N} \ell(U_n) y_n \right\| < \varepsilon. \tag{4.2}$$

Now let $P_0 = \{U_n : n = 1, ..., N_\varepsilon\}$. Then, clearly, $P_0 \in \Pi(A)$. Then, for every $P = \{(J_j, t_j) : j = 1, 2, ..., n_P\} \in \widetilde{\Pi}(A)$ such that $P \succ P_0$ there exists a finite subset $\{n_1, n_2, ..., n_k\}$ of $\mathbb{N} \setminus \{1, 2, ..., N_\varepsilon\}$ such that

$$\sum_{j=1}^{n_P} \ell(J_i) f(t_i) = \sum_{n \leq N_\varepsilon} \ell(U_n) y_{\sigma(n)} + \sum_{j=1}^{k} \ell(J_{n_j}) y_{n_j}.$$

It follows that

$$\left\| \sum_{j=1}^{n_P} \ell(J_i) f(t_i) - x \right\| \leq \left\| \sum_{n \leq N_\varepsilon} \ell(U_n) y_{\sigma(n)} - x \right\| + \left\| \sum_{j=1}^{k} \ell(J_{n_j}) y_{n_j} \right\|. \tag{4.3}$$

Let $c_j = \ell(J_{n_j}) / \ell(U_{n_j})$. Then $c_j \in (0, 1]$. We get from (4.2) that

$$\left\| \sum_{j=1}^{k} \ell(J_{n_j}) y_{n_j} \right\| \leq c_1 \left\| \sum_{j=1}^{k} \ell(J_{n_j}) y_{n_j} \right\| + | c_2 - c_1 | \left\| \sum_{j=1}^{k} \ell(J_{n_j}) y_{n_j} \right\|$$

$$+ \cdots + | c_k - c_{k-1} | \| \ell(J_{n_k}) y_{n_k} \| < \varepsilon. \tag{4.4}$$

By combining (4.1), (4.3) and (4.4), we conclude that

$$\left\| \sum_{j=1}^{n_P} \ell(J_i) f(t_i) - x \right\| < 2\varepsilon.$$

Thus, f is unconditionally ℓ-integrable over the set A. □

As for strong integrals, there have been many definitions for weak integrals. In what follows, we shall denote by X^* the continuous dual of the normed space *X*.

Definition 17. We say that a function $f : \Omega \to X$ is

(1) *scalarly* ℓ*-integrable* over a set $A \subset \Omega$ with respect to a given size function ℓ if for each $x^* \in X^*$, the numerical function

$$x^* f : \Omega \to [0, \infty) : \omega \mapsto x^* f(\omega)$$

is ℓ-integrable over the set *A*.

(2) *absolutely scalarly* ℓ*-integrable* over a set $A \subset \Omega$ with respect to a given size function ℓ if for each $x^* \in X^*$, the numerical function

$$| x^* f(\cdot) | : \Omega \to [0, \infty) : \omega \mapsto | x^* f(\omega) |$$

is ℓ-integrable over the set *A*.

We denote by $\mathcal{I}^*(A, \ell, X)$ (resp. $|\mathcal{I}^*|(A, \ell, X)$), the space of all scalarly (resp. absolutely scalarly) ℓ-integrable functions over the set *A*. It follows from the Dominated Convergence Theorem 13 that $|\mathcal{I}^*|(A, \ell, X) \subset \mathcal{I}^*(A, \ell, X)$.

Theorem 18. *If a function* $f \in \mathcal{I}^*(A, \ell, X)$, *then there exists a vector* $\int_A^{**} fd\ell \in X^{**}$ *such that for every* $x^* \in X^*$,

$$\left\langle x^*, \int_A^{**} fd\ell \right\rangle = \int_A x^* fd\ell.$$

Proof. We notice that for each $x^* \in X^*$, the map $\mu_{x^*} : B \mapsto \int_{A \cap B} x^* fd\ell$ for $B \in 2^\Omega$ defines an element of $M(\Omega)$, the Banach space of scalar measures (not necessarily non-negative) with the semi-variation norm. Consider the operator $T : X^* \to M(\Omega)$ defined by $Tx^* = \mu_{x^*}$. Then the adjoint T^* of T maps $M(\Omega)^*$ into X^{**}. For each $B \in 2^\Omega$, the indicator function 1_B defines an element of $M(\Omega)$ as follows: $\langle \mu, 1_B \rangle = \int_{A \cap B} d\mu$ for every $\mu \in M(\Omega)$. It follows in particular that $T^* 1_A \in X^{**}$ and we have for every $x^* \in X^*$,

$$\langle x^*, T^* 1_A \rangle = \langle Tx^*, 1_A \rangle = \langle \mu_{x^*}, 1_A \rangle = \int_A x^* f d\ell.$$

Hence, $\int_A^{**} f d\ell = T^* 1_A \in X^{**}$, as desired. □

If $f \in \mathcal{I}^*(A, \ell, X)$, then we call the vector $\int_A^{**} f d\ell \in X^{**}$ as the ℓ^{**}*-integral* of *f*. We say that *f* is ℓ*-Pettis integrable* if $\int_A^{**} f d\ell \in X$.

Note again that in any of the above definitions, ℓ is not necessarily a measure and no measurability assumptions are required. However, when strong measurability is assumed, then we have a quite interesting result. First, we say that a function $f : \Omega \to X$ is ℓ*-essentially separably valued* on a set *A* if there exists a separable subspace *Y* of *X* such that $\ell(\{\omega \in A : f(\omega) \notin Y\}) = 0$. Then we have the following theorem which can be seen as an extension of the Orlicz-Pettis Theorem.

Theorem 19. *Let X be a Banach space, $f : \Omega \to X$ be an ℓ-essentially separably valued function. Assume that f is $\Sigma(\ell)$-measurable, and let $A \in \Sigma(\ell)$ be of finite size. Then f is ℓ-Pettis-integrable over A if and only if f is ℓ-integrable over A; in which case,*

$$\int_A^{**} f d\ell = \int_A f d\ell.$$

Proof. The sufficiency is obvious. For the necessity, we notice that since *f* is ℓ-essentially separably valued on *A* and $\Sigma(\ell)$-measurable, *f* is the ℓ-limit of $\Sigma(\ell)$-simple functions, that is to say, f is measurable with respect to the measure restriction of ℓ to $\Sigma(\ell)$. The necessity then follows from Theorem 14 of [11].

To establish the equality, we notice that since the integral is a limit operator and since each $x^* \in X^*$ is continuous linear operator, we have

$$x^* \int_A f d\ell = \int_A x^* f d\ell = \left\langle x^*, \int_A^{**} f d\ell \right\rangle = x^* \int_A^{**} f d\ell,$$

and hence the desired equality. □

Corollary 20. *If X is a separable Banach space, then ℓ-Pettis-integrability is equivalent to ℓ-integrability over a measurable set A for (weakly) measurable functions.*

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