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Tensor Integral: A New Comprehensive Approach to the Integration Theory

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Abstract

We use tensor product to introduce a new approach to the theory of integration. Such an approach will strengthen the existing various classical concepts of integral and will provide a continuous thread tying the subject matter together. The integral of vector-valued functions with respect to vector-valued additive measures will be covered without any assumption of measurability. As applications, we state and prove extensions of the Lebesgue fundamental theorems of convergence in a more general setting.

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1 Introduction

Theories of vector integration of various kinds have been extensively studied and developped by many authors, owing to its successful application in many areas of functional analysis, probability theory, stochastic processes. For reference, we especially mention the books by Diestel and Uhl [1], Dinculeanu [2], Dunfordl and Schwartzl [3] and Kussmaul [4].

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The notion of integral introduced by the author in Robdera [5] provides an extension of all of the classical notions of integral of vector valued functions with respect to a scalar measure, namely, the Pettis-, Bochner-, McShane-, Henstock-Kurtzweil integrals of vector valued functions. However, such an extension does not cover the more general setting of the integral of scalar-valued functions with respect to vector measures, nor the more general case of the integral of vector-valued functions with respect to vector-valued additive measures.

The purpose of this note is to introduce, via tensor product, a new general and comprehensive approach to integration of vector-valued functions with respect to vector-valued additive measures, of which existing classical integration theories are all special cases. Such an extension will completely forgo any measurability assumption, allowing us for example to give simple proofs for the Lebesgue fundamental theorems of convergence in a more general setting.

2 Tensor Integral

Throughout this paper, V,W and U will denote normed spaces over the same scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $T : V \times W \to U$ is a continuous bilinear mapping such that for every $(v, w) \in V \times W$, the following inequality holds

$$\|v\|_V = \sup \left\{ \|T(v, w)\|_U : \|w\|_W \leq 1 \right\}$$

for every $v \in V$. We shall simply call such a mapping a **tensor**. For details on tensor product of Banach spaces, we refer the reader to Ryan [6].

Important examples of such bilinear mappings are:

1. the projective tensor product: $T(v, w) = v \otimes w$ from $V \times W \to V \hat{\otimes}_\pi W$.
2. the duality: $T(v, v') = \langle v, v' \rangle$ from $V \times V' \to \mathbb{K}$ where V' is the Banach dual of V, and where $\mathbb{K}$ is the scalar field.
3. If V is a Hilbert space, the inner product: $T(v, w) = \langle v, w \rangle$ from $V \times V \to \mathbb{K}$, where $\mathbb{K}$ is the scalar field.
4. The scaling of vectors: $T(\alpha, v) = \alpha v$ from $\mathbb{K} \times V \to V$, where $\mathbb{K}$ is the scalar field.

Note that since $\mathbb{K} \hat{\otimes}_\pi V \simeq V$, the scaling tensor in the above example 4 can actually be considered as the projective tensor product $T(\alpha, v) = \alpha v$ from $\mathbb{K} \times V \to V$.

Throughout the paper, Ω is a nonempty set, Σ is a ring of subsets of Ω, and $\mu : \Sigma \to W$ is an additive measure, that is, μ satisfies:

1. $\mu(\emptyset) = 0$;
2. $\mu(A \bigcup B) = \mu(A) + \mu(B)$ for every disjoint pair $(A, B) \in \Sigma \times \Sigma$.

By a ***Σ-subpartition*** of a set $A \in 2^\Omega$, we mean any finite collection

$$P = \{I_i : I_i \subset A, i = 1, 2 \ldots, n\} \subset \Sigma$$

satisfying

1. $\|\mu(I_i)\| < \infty$ for all i;
2. $I_i \cap I_j = \emptyset$ whenever $i \neq j$.

We denote by $\bigsqcup P$ the subset of A obtained by taking the union of all elements of P. A Σ-subpartition $P = \{I_i : i = 1, \ldots, n\}$ is said to be *tagged* if a point $t_i \in I_i$ is chosen for each $i \in \{1, \ldots, n\}$. We write $P := \{(I_i, t_i) : i \in \{1, \ldots, n\}\}$ if we wish to specify the tagging points. We denote by $\Pi(A, \Sigma)$ the collection of all tagged Σ-subpartitions of the set A. The *mesh* or the *norm* of $P \in \Pi(A, \Sigma)$ is defined to be

$$\|P\| = \max\{\|\mu(I_i)\| : I_i \in P\}.$$

If $P, Q \in \Pi(A, \Sigma)$, we say that Q is a **refinement** of P and we write $Q \succ P$ if $\|Q\| \leq \|P\|$ and $\bigsqcup P \subset \bigsqcup Q$. It is readily seen that such a relation does not depend on the tagging points. It is also easy to see that the relation $\succ$ is transitive on $\Pi(A, \Sigma)$. If $P, Q \in \Pi(A, \Sigma)$, we denote

$$P \vee Q := \{I \setminus J, I \cap J, J \setminus I : I \in P, J \in Q\}.$$

Clearly, $P \vee Q \in \Pi(A, \Sigma)$, $P \vee Q \succ P$ and $P \vee Q \succ Q$. Thus the relation $\succ$ has the upper bound property on $\Pi(A, \Sigma)$. We then infer that the set $\Pi(A, \Sigma)$ is directed (in the sense of Moore-Smith as described in McShane [7],) by the binary relation $\succ$.

Definition 2.1. Let V,W and U be Banach spaces, and let $T : V \times W \to U$ be a tensor. Let Ω be a nonempty set and $\Sigma \subset 2^{\Omega}$. Given a function $f : \Omega \to V$, and a tagged Σ-subpartition $P = \{(I_i, t_i) : i \in \{1, \ldots, n\}\}$, we define the **tensor Riemann sum** of f at P with respect to an additive measure $\mu : \Sigma \to W$ to be the element of U given by

$$f_\mu(P) = \sum_{i=1}^{n} T(f(t_i), \mu(I_i)).$$

Thus the function $P \mapsto f_\mu(P)$ is a U-valued net defined on the directed set $(\Pi(A, \Sigma), \succ)$. For convenience, we are going to denote the net-limit by

$$\int_A T(f, d\mu) := \lim_{(\Pi(A,\Sigma),\succ)} f_\mu(\cdot)$$

whether or not such a limit exists. For details on net-limit we refer the reader to McShane [7].

The notion of tensor integrability of a function with respect to a vector additive measure is defined as follows.

Definition 2.2. Let V,W and U be Banach spaces, and let $T : V \times W \to U$ be a tensor. Let Ω be a nonempty set and $\Sigma \subset 2^{\Omega}$ and let $\mu : \Sigma \to W$ be an additive measure. We say that a function $f : \Omega \to V$ is **Σ, T-integrable over a set A with respect to** μ (or **Σ, μ, T-integrable)** if the limit $\int_A T(f, d\mu)$ represents a vector in U. The vector $\int_A T(f, d\mu)$ is then called the **Σ, μ, T-integral** of f relative to μ over the set A.

In other words, $f : \Omega \to V$ is Σ, μ, T-integrable over the set A with Σ, μ, T-integral $\int_A T(f, d\mu)$ if for every $\epsilon > 0$, there exists a Σ-subpartition P_0 of the set A such that for every $P \succ P_0$ in $\Pi(A, \Sigma)$ we have

$$\left\| \int_A T(f, d\mu) - f_\mu(P) \right\|_U < \epsilon. \tag{2.1}$$

If $\Sigma = 2^{\Omega}$, we simply say that $f : \Omega \to V$ is μ, T-integrable over the set A.

Remark 2.1. Note that in the above definition, no notion of measurability is required.

We notice that the uniqueness of net-limit ensures us that there exists at most one vector $\int_A T(f, d\mu)$ that satisfies the property in Definition 2.2. We shall denote by $\mathcal{I}^T(A, \Sigma, \mu, V)$ the set of all functions $f : \Omega \to V$ that are Σ, μ, T-integrable over a given subset A of Ω. We also infer that being a limit operator, the tensor integral is linear, and that $\mathcal{I}^T(A, \Sigma, \mu, V)$ is a vector space. If $\Sigma = 2^{\Omega}$, we simply write $\mathcal{I}^T(A, \Sigma, \mu, V) = \mathcal{I}^T(A, \mu, V)$.

It is also clear that if A and B are disjoint subsets of Ω, then every subpartition R of the disjoint union $A \sqcup B$ is of the form $P \sqcup Q$ where $P \in \Pi(A)$ and $Q \in \Pi(B)$. It then follows that $f_\mu(R) = f_\mu(P) + f_\mu(Q)$. Thus if a function $f : \Omega \to V$ is μ-tensor integrable over both a set A and a set B, such that $A \cap B = \emptyset$, then f is μ-tensor integrable over the disjoint union $A \sqcup B$ and

$$\int_{A \sqcup B} T(f, d\mu) = \int_A T(f, d\mu) + \int_B T(f, d\mu).$$

We finish this section with few familiar examples. If $\mathbb{K}$ is the scalar field, it is wel known fact that $\mathbb{K}\hat{\otimes}_\pi W \simeq W$. Therefore the tensor integral with respect to the scaling of vectors $T(\alpha, w) = \alpha w$ from $\mathbb{K} \times W \to W$ reduces to the integral of scalar functions with respect to vector-valued additive measures. Likewise the tensor integral relative to a scaling of vectors $T(v, \alpha) = \alpha v$ $V \times \mathbb{R} \to V$, and if $\mu : \Sigma \to \mathbb{R}$ is a nonnegative additive measure, the above defined notion of tensor integrability coincides with the special case of the notion of the extended integrability of vector valued functions with respect to a monotonic, σ-subadditive, nonnegative set function introduced in Robdera [5]. It follows that all notions of the classical integrability of vector valued functions with respect to a scalar measure, namely, the Pettis-, Bochner-, vector-valued McShane-, vector-valued Henstock-Kurtzweil integrals are all special cases of tensor integrability relative to the scaling tensor.

3 Projective Tensor Integral

In this section, we shall focus on the special case of tensor integral with respect to the projective tensor product: $T(v, w) = v \otimes w$ from $V \times W \to V\hat{\otimes}_\pi W$. We shall denote by π the projective norm on $V\hat{\otimes}_\pi W$, and we denote the corresponding space of Σ-tensor integrable functions by $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$.

For every $f : \Omega \to V$, we define the Σ, μ-*variation* of f over the set $A \subset \Omega$ to be

$$\text{var}_{\Sigma,\mu}(f, A) := \sup \{\pi (f_\mu(P)) : P \in \Pi(A, \Sigma)\}.$$

We say that the function f is of *bounded* Σ, μ-*variation* if $\text{var}_\Sigma(f, A) < \infty$.

Since every convergent net is bounded, we notice that if $f \in \mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$ then f is of bounded Σ, μ-variation. We then define for $f \in \mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$

$$\|f\|_{\mathcal{I}} = \text{var}_{\Sigma,\mu}(f, A).$$

It is readily seen that $f \mapsto \|f\|_{\mathcal{I}}$ defines a seminorm on the space $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$.

Theorem 3.1. *Let V and W be Banach spaces, Ω a nonempty set and $\Sigma \subset 2^\Omega$. Let $\mu : \Sigma \to W$ be an additive measure such that $\|\mu(C)\|_W < \infty$ for all $C \in \Sigma$. Then the function space $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$ is complete with respect to the seminorm $\|\cdot\|_{\mathcal{I}}$.*

Proof. Let $n \mapsto f_n$ be a Cauchy sequence in $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$ with respect to the seminorm $\|\cdot\|_{\mathcal{I}}$. Fix $\epsilon > 0$, and choose $N_\epsilon > 0$ such that for $m, n > N_\epsilon$ in $\mathbb{N}$,

$$\|f_n - f_m\|_{\mathcal{I}} = \sup \{\pi ((f_n - f_m)_\mu(P)) : P \in \Pi(A, \Sigma)\} < \epsilon. \tag{3.1}$$

In particular, if we consider the subpartition $\{(A, \omega)\} \in \Pi(A, \Sigma)$, then for $m, n > N_\epsilon$ in $\mathbb{N}$,

$$\epsilon \geq \pi ((f_n(\omega) - f_m(\omega)) \otimes \mu(A)) = \|f_n(\omega) - f_m(\omega)\|_V \|\mu(A)\|_W .$$

We infer that the sequence $n \mapsto f_n(\omega)$ is Cauchy in V. Since V is a Banach space, we can define a function

$$\begin{array}{llll} f : & \Omega & \to & V \\ & \omega & \mapsto & \lim\limits_{n\to\infty} f_n(\omega) \end{array}$$

On the other hand, since $f_n, f_m \in \mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$, there exist $P_n, P_m \in \Pi(A, \Sigma)$ such that

$$\begin{aligned} \pi \left(f_{n,\mu}(P) - \int_A f_n \otimes d\mu \right) &< \epsilon \text{ whenever } P \succ P_n, \\ \pi \left(f_{m,\mu}(P) - \int_A f_m \otimes d\mu \right) &< \epsilon \text{ whenever } P \succ P_m. \end{aligned}$$

Combining these last two inequalities with (3.1), it follows that for $m, n > N_\epsilon$ in $\mathbb{N}$ and for every $P \succ P_n \vee P_m$, we have

$$\begin{aligned}\pi\left(\int f_n \otimes d\mu - \int_A f_n \otimes d\mu\right) \quad &\leq \quad \pi\left(f_{n,\mu}(P) - \int_A f_n \otimes d\mu)\right) \\ &\quad +\pi\left((f_n - f_m)_\mu(P)\right) \\ &\quad +\pi\left(f_{m,\mu}(P) - \int_A f_m \otimes d\mu)\right) < 3\epsilon.\end{aligned}$$

This proves that the sequence $n \mapsto \int_A f_n \otimes d\mu$ is Cauchy in the Banach space $V\hat{\otimes}_\pi W$, and thus converges to some element, say $a \in V\hat{\otimes}_\pi W$.

Now let $P = \{(I_i, t_i) : i \in \{1, \ldots, k\}\} \in \Pi(A, \Sigma)$. For each $t_i \in I_i$, there exists $N_i > N_\epsilon$ such that for $m, n > N_i$ in $\mathbb{N}$,

$$\|f_n(t_i) - f_m(t_i)\|_V \|\mu(I_i)\|_W = \pi\left((f_n(t_i) - f_m(t_i)) \otimes \mu(I_i))\right) \leq \frac{\epsilon}{2^i}.$$

It follows that for $m, n > \max\{N_i : i = 1, \ldots, k\} =: N_P$, we have

$$\pi\left((f_n - f_m)_\mu(P)\right) \leq \sum_{i=1}^{k} \|f_n(t_i) - f_m(t_i)\|_V \|\mu(I_i)\|_W \leq \epsilon.$$

If we let $m \to \infty$, we obtain $\pi\left((f_n - f)_\mu(P)\right) \leq \epsilon$.

On the other hand, since $a = \lim_{n\to\infty} \int_A f_n \otimes d\mu$, there exists $N > N_P$ such that for $m > N$

$$\pi\left(\int_A f_m \otimes d\mu - a\right) < \epsilon.$$

Thus for $n, m > N$,

$$\begin{aligned}\pi\left(f_\mu(P) - a\right) \quad &\leq \quad \pi\left((f - f_n)_\mu(P)\right) + \pi\left((f_n - f_m)_\mu(P)\right) \\ &\quad +\pi\left(\int_A f_m \otimes d\mu - a\right) < 3\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this shows that $f \in \mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$ and that $\int_A f \otimes d\mu = a$. □

It should be clear that if the set $C \in \Sigma$ is such that $\mu(C) = 0$, then for all subpartitions $P \in \Pi(C)$, $f_\mu(P) = 0$, and thus $\int_C f \otimes d\mu = 0$. It follows that the tensor-integral does not distinguish between functions which differ only on set $C \in \Sigma$ such that $\mu(C) = 0$. To make this more precise,

$$\int_A f \otimes d\mu = \int_A g \otimes d\mu \quad \text{whenever} \quad \mu\{x \in A : f(x) \neq g(x)\} = 0.$$

We say that a function f is essentially equal on A to another function g, and we write $f \sim g$ if $\mu\{x \in A : f(x) \neq g(x)\} = 0$. It is readily seen that the relation $f \sim g$ is an equivalence relation on $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)$. We shall denote by $I^{\otimes}(A, \Sigma, \mu, V)$ the quotient space $\mathcal{I}^{\otimes}(A, \Sigma, \mu, V)/\sim$. The restriction of the seminorm $\|\cdot\|_{\mathcal{I}}$ is a norm on $I^{\otimes}(A, \Sigma, \mu, V)$ that makes it a Banach space.

4 Fundamental Theorems of Convergence

In this section, we extend and prove the Lebesgue convergence theorems in the setting of tensor integral with respect to an additive measure taking values in a Banach lattice.

We assume that the Banach space W is equipped with an order relation $\leq$ that will make it a Banach lattice. We consider a nonnegative additive measure $\mu : \Sigma \to W$. The following properties are readily verified.

- μ is monotonic, that is $\mu(C) \leq \mu(B)$ whenever $C \subset B$ in Σ.
- μ is countably subadditive, that is, $\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$ whenever $A_n \in \Sigma$ are such that $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

Throughout this section, we shall consider tensor integral relative to the scaling tensor $\tau(\alpha, w) = \alpha w$ from $\mathbb{K} \times W \to W$ and relative to an additive lattice measure $\mu : \Sigma \to W$.

Fatou's Lemma

We say that a function $f : \Omega \to V$ is norm Σ, μ, τ-integrable over a set $A \subset \Sigma$ if the function $\|f\|_V$ is integrable Σ, μ, τ-integrable over A.

Theorem 4.1. *Let $f_n : \Omega \to V$ be a sequence of norm Σ, μ, τ-integrable functions over a set $A \subset \Sigma$ such that for every $\omega \in A$, $f(\omega) := \liminf_{n\to\infty} \|f_n(\omega)\|_V$. Then*

$$\int_A f d\mu \leq \liminf_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu. \tag{4.1}$$

Proof. We first note that the tensor integrals on the left hand side and the right hand side in (4.1) are both valued in the Banach lattice $(W, \leq)$ and the inequality is in the sense of the lattice structure of of W.

By the definition of the tensor integral, we are done if we show that

$$f_\mu(P) \leq \liminf_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu$$

for all $P = \{(I_i, t_i) : i = 1, \dots, m\} \in \Pi(A)$.

Fix $P \in \Pi(A, \Sigma)$. Define $\varphi_P(\omega) = \sum_{I_i\in P} 1_I(\omega) f(t_i)$. We notice that $\varphi_{P,\mu}(P) = f_\mu(P)$. We let

$$\begin{aligned} m &= \min\{\varphi_P(\omega) : \omega \in A\} \\ M &= \max\{\varphi_P(\omega) : \omega \in A\}, \end{aligned}$$

and we define

$$E = \{\omega \in A : \varphi_P(\omega) > m\}.$$

We notice that $m\mu(E) \leq \varphi_{P,\mu}(P)$ in the Banach lattice W. Fix $\epsilon > 0$ and define

$$E_n = \{\omega \in A : \|f_k(\omega)\|_V > (1 - \epsilon)\varphi_P(\omega), \ \forall k \geq n\}.$$

Then $E \subset \bigcup_n E_n$, and $E_n \subset E_{n+1}$ for all n. We have

$$\lim_{n\to\infty} \mu(E \setminus E_n) = \mu(\emptyset) = 0.$$

Thus we can choose an integer n_0 such that $\|\mu(E \setminus E_n)\|_W < \epsilon$ for all $n > n_0$. Thus if $n > n_0$, we have

$$\int_A \|f_n(\cdot)\|_V \, d\mu \geq \int_{E_n} \|f_n(\cdot)\|_V \, d\mu \geq (1 - \epsilon) \int_{E_n} \varphi_P(\omega) d\mu.$$

On the other hand,

$$\int_A \varphi_P d\mu = \int_E \varphi_P d\mu = \int_{E_n} \varphi_P d\mu + \int_{E\setminus E_n} \varphi_P d\mu.$$

Hence

$$\begin{aligned}
\int_A \|f_n(\cdot)\|_V \, d\mu &\geq (1-\epsilon)\int_{E_n} \varphi_P(\omega) d\mu \\
&= (1-\epsilon)\left[\int_A \varphi_P d\mu - \int_{E\setminus E_n} \varphi_P d\mu\right] \\
&\geq (1-\epsilon)\left[\int_A \varphi_P d\mu - M\mu(E\setminus E_n)\right] \\
&= \int_A \varphi_P d\mu - \epsilon\left[\int_A \varphi_P d\mu + M\mu(E\setminus E_n)\right].
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get that

$$\liminf_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu \geq \int_A \varphi_P d\mu = f_\mu(P).$$

The proof is complete. □

Monotone Convergence Theorem

We first notice that if $\mu : \Sigma \to W$ is an additive measure, then we have the following

Proposition 4.1. *Let $f,g : \Omega \to \mathbb{R}$ be both Σ,μ,τ-integrable over a A, and $h : \Omega \to \mathbb{R}$ such that $f(\omega) \leq h(\omega) \leq g(\omega)$, for all $\omega \in A$ then*

1. *h is Σ,μ,τ-integrable and*
2. *$\int_A f d\mu \leq \int_A h d\mu \leq \int_A g d\mu$.*

Proof. It suffices to notice that for all $P \in \Pi(A,\Sigma)$, one has $f_\mu(P) \leq f_\mu(P) \leq f_\mu(P)$. □

In particular, we infer from the above proposition that the tensor-integral with respect to an additive lattice measure is monotonic.

Theorem 4.2. *Let $f_n : \Omega \to V$ be a sequence of norm Σ,μ-tensor integrable functions satisfying:*

1. *$0 \leq \|f_n(\omega)\|_V \leq \|f_{n+1}(\omega)\|_V$, for every $\omega \in A \subset \Omega$ and for all $n \in \mathbb{N}$;*
2. *for every $\omega \in A$, $f(\omega) := \lim_{n\to\infty} f_n(\omega)$;*

Then $\|f\| : A \to \mathbb{R}$ is Σ,τ-integrable with respect to μ if and only if $\lim\limits_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu$ exists in W. Moreover,

$$\int_A \|f(\cdot)\|_V \, d\mu = \lim_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu.$$

Proof. It follows from the monotonicity property of the tensor integral (Proposition 4.1) that

$$\int_A \|f_n(\cdot)\|_V \, d\mu \leq \int_A \|f_{n+1}(\cdot)\|_V \, d\mu \leq \int_A \|f(\cdot)\|_V \, d\mu.$$

Hence the sequence $n \mapsto \int_A \|f_n(\cdot)\|_V \, d\mu$ is a non-decreasing net of vectors in W. On the one hand, if f is Σ,μ-tensor integrable, that is, if $\int_A \|f(\cdot)\|_V \, d\mu$ exists as a vector in W then for every n

$$\int_A \|f_n(\cdot)\|_V \, d\mu \leq \int_A \|f(\cdot)\|_V \, d\mu.$$

On the other hand, if $\lim_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu$ exists in W, then by Fatou's Lemma, we have

$$\int_A \|f(\cdot)\|_V \, d\mu \leq \liminf_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu \leq \lim_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu.$$

In both cases, we have $\lim_{n\to\infty} \int_A \|f_n(\cdot)\|_V \, d\mu = \int_A \|f(\cdot)\|_V \, d\mu.$ □

Dominated Convergence Theorem

Again $\mu : \Sigma \to W$ is an additive measure.

Theorem 4.3. *Let $f_n : \Omega \to V$ be a sequence of norm Σ, μ, τ-integrable functions satisfying the following properties:*

1. $f_n(\omega) \to f(\omega)$ *for all* $\omega \in A \subset \Omega$;
2. *there exists a real valued function* $h \in \mathcal{I}(A, \Sigma, \mu)$ *such that* $\|f_n(\omega)\|_V \leq h(\omega)$ *for all* $\omega \in A$, *and for all* $n \in \mathbb{N}$.

Then

1. f *is norm* Σ, μ, τ*-ntegrable;*
2. $\lim_{n\to\infty} \int_A \|f - f_n\|_V \, d\mu = 0$*;*
3. $\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu.$

Proof. It follows from the conditions of the theorem that for all $\omega \in A$, and for all $n \in \mathbb{N}$, we have

$$\|f(\omega) - f_n(\omega)\|_V \leq 2h(\omega)$$

and $\limsup_{n\to\infty} \|f(\omega) - f_n(\omega)\| = 0$ for each $\omega \in A$. Using the linearity and the monotonicity of the integral, we get that

$$\pi\left(\int_A f d\mu - \int_A f_n d\mu\right) = \pi\left(\int_A (f - f_n) d\mu\right) \leq \int_A \|f - f_n\|_V \, d\mu.$$

By Fatou's Lemma, we have

$$\limsup_{n\to\infty} \int_A \|f - f_n\|_V \, d\mu \leq \int_A \limsup_{n\to\infty} \|f - f_n\|_V \, d\mu = 0.$$

The result follows. □

It is easily noticed that for the special case where Σ is a σ-algebra and $\mu : \Sigma \to \mathbb{R}$ is an Lebesgue measure on Σ, each one of the above theorems reduces to the usual Lebesgue convergence theorems.

5 Conclusions

We have introduced a new comprehensive approach to integration theory.

a In Section (2), the notion of tensor integral of Banach space valued functions has been introduced.

b In Section (3), the special case of the projective tensor integral is discussed.

c In Section (4), we stated and proved extensions of the Lebesgue convergence theorems.

Competing Interests

The author declares that no competing interests exist.

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