THEORY PROBAB. APPL.
Vol. 50, No. 4, pp. 718–722

ON ALMOST SURE BEHAVIOR OF STABLE SUBORDINATORS OVER RAPIDLY INCREASING SEQUENCES*

R. VASUDEVA† AND G. DIVANJI‡

Abstract. Let $(X(t),\ t \geqq 0)$ with $X(0) = 0$ be a stable subordinator with index $0 < \alpha < 1$ and let (t_k) be an increasing sequence such that $t_{k+1}/t_k \to \infty$ as $k \to \infty$. Let (a_t) be a positive nondecreasing function of t such that $a(t)/t \leqq 1$. Define $Y(t) = X(t + a(t)) - X(t)$ and $Z(t) = X(t) - X(t - a(t))$, $t > 0$. We obtain law-of-the-iterated-logarithm results for $(X(t_k)), (Y(t_k))$ and $Z(t_k)$, properly normalized.

Key words. law of iterated logarithm, subsequences, stable subordinators, almost sure bounds

DOI. 10.1137/S0040585X97982128

1. Introduction.

Let $W(t)$, $t \geqq 0$, denote a standard Wiener process. If (t_k) is such that $\limsup_{k\to\infty}(t_{k+1}/t_k) < \infty$, then proceeding as in [1], one can show that

$$\limsup_{k\to\infty} \frac{W(t_k)}{\sqrt{2t_k \log\log t_k}} = 1 \quad \text{a.s.}$$

But for some sequences (t_k) with $t_{k+1}/t_k \to \infty$ we have $\lim_{k\to\infty}(W(t_k)/\sqrt{2t_k \log\log t_k}) = 0$ a.s. In such cases the normalizing sequence $\sqrt{2t_k \log\log t_k}$ will not be precise enough to give a.s. bounds for $W(t_k)$. In general whenever $t_{k+1}/t_k \to \infty$, Schwabe and Gut [5] have pointed out that $\sqrt{2t_k \log\log t_k}$ is no longer the proper normalizing sequence and it has to be replaced by $\sqrt{2t_k \log k}$. (These results have been obtained by the above authors for partial sums of independent and identically distributed random variables with finite variance.) This observation motivated us to examine whether similar things happen in the case of stable subordinators. The answer turns out to be affirmative, as established in the next section.

We first present the following result of [6] on the behavior of the limit supremum of $(X(t_k)/t_k^{1/\alpha})^{1/\log\log t_k}$ and limit infimum of $(X(t_k)/(t_k^{1/\alpha}(\log\log t_k)^{(\alpha-1)/\alpha}))$ for sequences (t_k), when (t_k) is at most geometrically increasing and when (t_k) is at least geometrically increasing. The case $t_{k+1}/t_k \to \infty$ comes under the class of at-least-geometrically-increasing sequences.

Theorem A. *Let $X(t)$ denote a stable subordinator with index α, $0 < \alpha < 1$. Define $\theta_\alpha = \alpha(1-\alpha)^{(1-\alpha)/\alpha}(\cos(\pi\alpha/2))^{-1/\alpha}$. If $\limsup_{k\to\infty}(t_{k+1}/t_k) < \infty$, then*

$$\limsup_{k\to\infty}\left(\frac{X(t_k)}{t_k^{1/\alpha}}\right)^{1/\log\log t_k} = e^{1/\alpha} \quad \textit{a.s.} \quad \textit{and} \quad \liminf_{k\to\infty}\frac{X(t_k)}{t_k^{1/\alpha}(\log\log t_k)^{(\alpha-1)/\alpha}} = \theta_\alpha \quad \textit{a.s.}$$

If $\liminf_{k\to\infty} t_{k+1}/t_k > 0$, then

$$\limsup_{k\to\infty}\left(\frac{X(t_k)}{t_k^{1/\alpha}}\right)^{1/\log\log t_k} = e^{\lambda/\alpha} \quad \textit{a.s.},$$

where $\lambda = \inf\{\varepsilon > 0\colon \sum_{k=k_0}^{\infty}(\log t_k)^{-\varepsilon} < \infty\}$, and

$$\liminf_{k\to\infty}\frac{X(t_k)}{t_k^{1/\alpha}(\log\log t_k)^{(\alpha-1)/\alpha}} = \lambda\theta_\alpha \quad \textit{a.s.},$$

*Received by the editors September 3, 2003. This work was supported by the UGC–Departmental Research Support Grant and by the CSIR Major Research Project (CSIR letter 25(0117)/01/EMR-II, dated 12.03.2001).

http://www.siam.org/journals/tvp/50-4/98212.html

†Department of Statistics, University of Mysore, Manasagangothri, Mysore, 570 006, India.

‡Department of Statistics, Sri Krishnadevaraya University, Anantapur, 515 003, India; and Department of Statistics, University of Botswana, Botswana (gootydivan@yahoo.co.in).

where $\lambda = \inf\{\varepsilon > 0\colon \sum_{k=k_0}^{\infty} (\log t_k)^{\beta} < \infty,\ \beta = \varepsilon^{\alpha/(\alpha-1)}\}$.

Remark. When (t_{k+1}/t_k) is at most geometrically fast, we notice that both the limit supremum and the limit infimum remain unchanged, whereas when (t_{k+1}/t_k) is more than geometrically fast, the limit supremum and the limit infimum both change depending on the speed of (t_k). The limit supremum result depends on the right (heavy) tail of the process, whereas the limit infimum result depends on the tail near zero, which is exponentially fast. When $t_k = 2^{2^{\cdot^{\cdot^{2}}}}$ (k times) one can show that the limit supremum becomes 1 (the limit infimum of the same function is always 1 for any (t_k)) and the limit infimum (or limit) becomes ∞. In [6] it has been observed that over any $(t_k) \uparrow \infty$,

$$\liminf_{k\to\infty} \left(\frac{X(t_k)}{t_k^{1/\alpha}}\right)^{1/\log\log t_k} = 1 \quad \text{a.s.} \quad \text{and} \quad \limsup_{k\to\infty} \frac{X(t_k)}{t_k^{1/\alpha}(\log\log t_k)^{(\alpha-1)/\alpha}} = \infty \quad \text{a.s.}$$

In [6] it was also shown that for certain sequences (t_k) that are faster than geometric, the iterated logarithm results can still be obtained by replacing "log log" with "log log log." By following [5], we now obtain a.s. results with $\log k$ in place of $\log\log t_k$ whenever $t_{k+1}/t_k \to \infty$. We also obtain similar results for $(Y(t_k))$ and $(Z(t_k))$.

2. Main results.

THEOREM 1. *Let* (t_k) *be such that* $t_{k+1}/t_k \to \infty$ *as* $k \to \infty$. *Then*

$$\text{(A)} \qquad \limsup_{k\to\infty} \left(\frac{X(t_k)}{t_k^{1/\alpha}}\right)^{1/\log k} = e^{1/\alpha} \quad a.s.$$

Moreover, if $t_{k+1}/t_k \geqq k^{(1+\delta)}$, *for some* $\delta > 0$, *then*

$$\text{(B)} \qquad \liminf_{k\to\infty} \frac{X(t_k)}{t_k^{1/\alpha}(\log k)^{(\alpha-1)/\alpha}} = \theta_\alpha \quad a.s.$$

Proof. We first establish (A). Observe that $t^{-1/\alpha}X(t) = X(1)$, in distribution, where $X(1)$ is a positive stable random variable with index α. Hence for any given $\varepsilon > 0$, by the tail behavior of $X(1)$, we have

$$\mathbf{P}\big\{X(t_k) > t_k^{1/\alpha} k^{(1+\varepsilon)/\alpha}\big\} < Ck^{-(1+\varepsilon)}.$$

By the Borel–Cantelli lemma,

$$\text{(1)} \qquad \mathbf{P}\big\{X(t_k) > t_k^{1/\alpha} k^{(1+\varepsilon)/\alpha} \text{ i.o.}\big\} = 0$$

(here "i.o." stands for "infinitely often"). Also, using the fact that $t_{k+1}/t_k \to \infty$ as $k \to \infty$, one can show that for large k, $(X(t_{k+1}) - X(t_k))/t_k^{1/\alpha}$ is distributionally equivalent to $X(1)$, and hence

$$\mathbf{P}\big\{X(t_k) - X(t_{k-1}) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha}\big\} \geqq Ck^{-(1-\varepsilon)}.$$

Since $(X(t_k) - X(t_{k-1}))$ are mutually independent and $\sum_{k=1}^{\infty} k^{-(1-\varepsilon)} = \infty$, by again applying the Borel–Cantelli lemma one gets that

$$\text{(2)} \qquad \mathbf{P}\big\{X(t_k) - X(t_{k-1}) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha} \text{ i.o.}\big\} = 1.$$

From the fact that $X(t)$ is increasing, we have $X(t_k) > X(t_k) - X(t_{k-1})$. Hence (2) implies that

$$\text{(3)} \qquad \mathbf{P}\big\{X(t_k) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha} \text{ i.o.}\big\} = 1.$$

Now (1) and (3) together establish (A) of the theorem.

For any $\varepsilon > 0$, we have

$$\mathbf{P}\big\{t_k^{-1/\alpha} X(t_k) < (1-\varepsilon)\,\theta_\alpha (\log k)^{(\alpha-1)/\alpha}\big\} \leqq C(\log k)^{-(1-\alpha)/2\alpha} \exp\big\{-(1+\varepsilon_1)\log k\big\}$$
$$= C(\log k)^{-(1-\alpha)/\alpha} k^{-(1+\varepsilon)}$$

for some $\varepsilon_1 > 0$. Hence by the Borel–Cantelli lemma, one gets

$$\mathbf{P}\big\{t_k^{-1/\alpha} X(t_k) < (1-\varepsilon)\,\theta_\alpha (\log k)^{-(1-\alpha)/\alpha} \text{ i.o.}\big\} = 0. \tag{4}$$

Again recalling that for large k, $t_k^{-1/\alpha}(X(t_k) - X(t_{k-1}))$ has the same distribution as $X(1)$, one gets

$$\mathbf{P}\Big\{t_k^{-1/\alpha}\big(X(t_k) - X(t_{k-1})\big) < (1+\varepsilon)\,\theta_\alpha (\log k)^{-(1-\alpha)/\alpha}\Big\} \geqq Ck^{-(1-\varepsilon)}.$$

By the Borel–Cantelli lemma, we have

$$\mathbf{P}\Big\{t_k^{-1/\alpha}\big(X(t_k) - X(t_{k-1})\big) < (1+\varepsilon)\,\theta_\alpha (\log k)^{-(1-\alpha)/\alpha} \text{ i.o.}\Big\} = 1. \tag{5}$$

Also,

$$\mathbf{P}\big\{t_k^{-1/\alpha} X(t_{k-1}) > \varepsilon\,\theta_\alpha (\log k)^{-(1-\alpha)/\alpha}\big\}$$
$$\leqq \mathbf{P}\big\{t_{k-1}^{-1/\alpha} X(t_{k-1}) > \varepsilon\, t_k^{1/\alpha} t_{k-1}^{-1/\alpha} \theta_\alpha (\log k)^{-(1-\alpha)/\alpha}\big\} \leqq Ck^{-(1+\delta)}$$

for some $\delta > 0$. By the Borel–Cantelli lemma, one gets

$$\mathbf{P}\big\{t_k^{-1/\alpha} X(t_{k-1}) > \varepsilon\,\theta_\alpha (\log k)^{-(1-\alpha)/\alpha} \text{ i.o.}\big\} = 0. \tag{6}$$

Relations (4), (5), and (6) together complete the proof of (B).

3. Boundary crossings associated with the law of the iterated logarithm. Define, for any $\varepsilon > 0$,

$$U_k = \begin{cases} 1 & \text{if } \left(\dfrac{X(t_k)}{t_k^{1/\alpha}}\right)^{1/\log k} > e^{(1+\varepsilon)/\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $N_\varepsilon = \sum_{k=1}^\infty U_k$. Note that N_ε is a proper random variable giving the number of boundary crossings of $(X(t_k)/t_k^{1/\alpha})^{1/\log k}$. Then $\mathbf{E}N_\varepsilon = \sum_{k=1}^\infty \mathbf{P}\{U_k = 1\} \leqq \sum_{k=1}^\infty k^{-(1+\varepsilon)} < \infty$. Hence the expected number of boundary crossings is finite. Similarly, if N_ε^* is the number of crossings of the lower boundary $(1-\varepsilon)\,\theta_\alpha$ of the sequence $(X(t_k)/t_k^{1/\alpha}(\log k)^{(\alpha-1)/\alpha})$, then one can show that $\mathbf{E}N_\varepsilon^* < \infty$.

We now discuss the behavior of $(Y(t_k))$ when (t_k) satisfies $t_{k+1}/t_k \to \infty$ as $k \to \infty$. First we present the following known result for $(Y(t))$ as $t \to \infty$.

Theorem B. *Let $d(t) = \log(t/a_t) + \log\log t$. Then*

$$\mathbf{P}\left\{\limsup \left(\frac{Y(t)}{a_t^{1/\alpha}}\right)^{1/d(t)} = e^{1/\alpha}\right\} = 1 \quad \textit{and} \quad \mathbf{P}\left\{\liminf \left(\frac{Y(t)}{a_t^{1/\alpha}}\right)^{1/d(t)} = 1\right\} = 1.$$

For the proof see [6].

Remark. Let $t_k = \rho^k$, $\rho > 1$. When $a_t = t^{1/2}$, we note that $d(t) \sim (\log t)/2$ and $\lim_{k\to\infty}(Y(t_k)/a_{t_k}^{1/\alpha})^{1/d(t_k)} = 1$ a.s. But if $a_t = t/2$, then

$$d(t) \sim \log\log t \quad \text{and} \quad \limsup_{k\to\infty} \left(\frac{Y(t_k)}{a_{t_k}^{1/\alpha}}\right)^{1/d(t_k)} = e^{1/\alpha} \quad \text{a.s.}$$

Hence for a geometrically increasing (t_k), the behavior of $(Y(t_k))$ changes with the form of a_t. However, for (t_k) such that $t_{k+1}/t_k \to \infty$ as $k \to \infty$ we have a unified result, as presented in what follows.

THEOREM 2. *Let (t_k) be such that $t_{k+1}/t_k \to \infty$ as $k \to \infty$. Then*

$$\mathbf{P}\left\{\limsup\left(\frac{Y(t_k)}{a_{t_k}^{1/\alpha}}\right)^{1/\log k} = e^{1/\alpha}\right\} = 1 \quad \textit{and} \quad \mathbf{P}\left\{\liminf\left(\frac{Y(t_k)}{a_{t_k}^{1/\alpha}}\right)^{1/\log k} = 1\right\} = 1.$$

Further, all points in $[1, e^{1/\alpha}]$ are a.s. limit points of $(Y(t_k)/a_{t_k}^{1/\alpha})^{1/\log k}$.

Proof. From the fact that $a_t < t$, note that $t_{k+1}/t_k \to \infty$ as $k \to \infty$ implies that $t_{k+1}/(t_k + a_{t_k}) \to \infty$ as $k \to \infty$, and hence for k large $(Y(t_k))$ are mutually independent. From the fact that $Y(t)/a_t^{1/\alpha}$ is distributionally the same as $X(1)$, the limit supremum and limit infimum results in the theorem follow by a straightforward application of the Borel–Cantelli lemma.

A point e^p, belonging to $[1, e^{1/\alpha}]$, is a limit point if for any $\varepsilon > 0$,

$$\mathbf{P}\left\{k^{(p-\varepsilon)/\alpha} < \frac{Y(t_k)}{a_{t_k}^{1/\alpha}} < k^{(p+\varepsilon)/\alpha} \quad \text{i.o.}\right\} = 1.$$

From the fact that

$$\mathbf{P}\left\{k^{(p-\varepsilon)/\alpha} < \frac{Y(t_k)}{a_{t_k}^{1/\alpha}} < k^{(p+\varepsilon)/\alpha}\right\} \geqq k^{-(p-\varepsilon)} \quad \text{and} \quad \sum_{k=1}^{\infty} k^{-(p-\varepsilon)} = \infty,$$

applying the Borel–Cantelli lemma we note that e^p is an a.s. limit point of $(Y(t_k)/a_{t_k}^{1/\alpha})^{1/\log k}$. The proof of the theorem is complete.

THEOREM 3. *Let (t_k) be such that $t_{k+1}/t_k \to \infty$ as $k \to \infty$. Then*

$$\mathbf{P}\left\{\limsup_{k\to\infty} \frac{Y(t_k)}{a_{t_k}^{1/\alpha}(\log k)^{(\alpha-1)/\alpha}} = \infty\right\} = 1 \quad \textit{and} \quad \mathbf{P}\left\{\liminf_{k\to\infty} \frac{Y(t_k)}{a_{t_k}^{1/\alpha}(\log k)^{(\alpha-1)/\alpha}} = \theta_\alpha\right\} = 1.$$

The proof follows along the lines of Theorem 2, but by considering the tail behavior near zero. The details are omitted.

Define

$$Q_k = \frac{1}{a_{t_k}^{1/\alpha}}\big((Y(t_k))^{1/\log k},\ (Z(t_k))^{1/\log k}\big), \quad R_k = \frac{1}{a_{t_k}^{1/\alpha}(\log k)^{(\alpha-1)/\alpha}}\big(Y(t_k),\ Z(t_k)\big)$$

and introduce the sets $S_1 = \{(x,y)\colon x \geqq 0,\, y \geqq 0,\, xy \leqq e^{1/\alpha}\}$ and $S_2 = \{(x,y)\colon x \geqq 0,\, y \geqq 0,\, xy \geqq \theta_\alpha\}$. Then we have the following result.

THEOREM 4. *Let $t_{k+1}/t_k \to \infty$ as $k \to \infty$. Then set of all a.s. limit points of (Q_k) coincides with S_1, and that of (R_k) coincides with S_2.*

Proof. For k large, $(Y(t_k), Z(t_k))$ becomes a mutually independent sequence with independent components. The proof follows along the lines of [4]. The details are omitted.

Acknowledgments. The authors express their sincere thanks to University Grants Commission, New Delhi, and Council of Scientific and Industrial Research, New Delhi, respectively, for their financial support of this work. The authors also thank the referee for his positive response in this work.

REFERENCES

[1] A. GUT, *Law of the iterated logarithm for subsequences*, Probab. Math. Statist., 7 (1986), pp. 27–58.

[2] J. Chover, *A law of the iterated logarithm for stable summands*, Proc. Amer. Math. Soc., 17 (1966), pp. 441–443.

[3] J. L. Mijnheer, *Sample Path Properties of Stable Processes*, Mathematisch Centrum, Amsterdam, 1975.

[4] R. P. Pakshirajan and R. Vasudeva, *A law of the iterated logarithm for stable summands*, Trans. Amer. Math. Soc., 232 (1977), pp. 33–42.

[5] R. Schwabe and A. Gut, *On the law of the iterated logarithm for rapidly increasing subsequences*, Math. Nachr., 178 (1996), pp. 309–320.

[6] R. Vasudeva, *Some almost sure results for stable subordinators*, in Proceedings of the International Conference on Stochastic Processes (Cochin, India, 1996), A. Krishnamoorphy, ed., 1996, pp. 125–130.